Autour de l'outil statistique Copule et tests non-paramétriques de détection de rupture dans la dépendance entre les composantes d'observations multivariées

–Animation scientifique GenPhySE–

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November 25, 2019

Formation

2006-2011

Licence mathématiques, Université de Strasbourg
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Doctorat de mathématique, Université de Pau, Université de Sherbrooke
▷ Deux tests de détection de rupture dans la copule d'observations multivariées
co-direction: Ivan Kojadinovic, Jean-François Quessy

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Parcours professionnel I

2014-2015

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Keywords:

- non-parametric statistic
- copula
- empirical processes
- multipliers bootstrap

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Parcours professionnel II

2016- nov 2018

Post-doctorat, Le Mans Université, PANORisk

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Rohmer, T., Dutang, C., Brouste, A.
Maximum likelihood estimation for generalized linear model in the case of categorical explanatory variables.
Soumission prochaine sur CRAN + article in JSS

Parcours professionnel III

2018- 2019

Post-doctorat, Inria, Ecole Polytechnique Paris-Saclay

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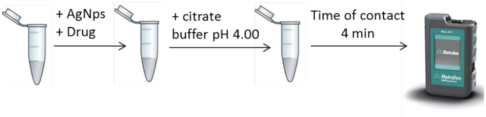

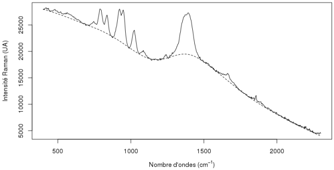

Summary

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Test for break detection

Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be d-dimensional random vectors, where for $i = 1, \ldots, n$, $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{id})$. We aim at testing the null hypothesis

$$\mathcal{H}_0 : \exists\, F \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have c.d.f. } F.$$

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Example of alternative hypotheses:

- An abrupt change at the instant $k^\star \in [\![1, n-1]\!]$:

$$\exists F^{(1)}, F^{(2)} \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_{k^\star} \sim F^{(1)}, \quad \boldsymbol{X}_{k^\star+1}, \ldots, \boldsymbol{X}_n \sim F^{(2)}$$

Test for break detection

Let $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ be d-dimensional random vectors, where for $i = 1, \dots, n$, $\boldsymbol{X}_i = (X_{i1}, \dots, X_{id})$. We aim at testing the null hypothesis

$$\mathcal{H}_0 : \exists\, F \text{ such that } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n \text{ have c.d.f. } F.$$

Example of alternative hypotheses:

- A gradual change: $\exists 1 \leq k_1 \leq k_2 \leq n-1$
  $\boldsymbol{X}_1, \dots, \boldsymbol{X}_{k_1} \sim F^{(1)}$
  $\boldsymbol{X}_{k_2}, \dots, \boldsymbol{X}_n \sim F^{(2)}$
  For $i = k_1 + 1, \dots, k_2$ the law of $\boldsymbol{X}_i$ will gradually go from $F^{(1)}$ to $F^{(2)}$.

Cumulative Sum test for $\mathcal{H}_0$, serially independent data I

Example 1: Test for change in the mean, d=1

$$T_n^\mu = \max_{k=1,\ldots,n-1} \frac{k(n-k)}{n^{3/2}} \left| \left\{ \frac{1}{k} \sum_{i=1}^{k} X_i - \frac{1}{n-k} \sum_{i=k+1}^{n} X_i \right\} \right|$$

$$= \sup_{s\in[0,1]} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} (X_i - \bar{X}_n) \right|.$$

- Under $\mathcal{H}_0$, as soon as $X_1, \ldots, X_n$ are i.i.d.,

$$T_n^\mu \rightsquigarrow \sigma \sup_{s\in[0,1]} |\mathbb{U}(s)|,$$

where σ^2 is the unknown variance of X_i and $\mathbb{U}$ is a standard Brownian bridge, i.e. a centered Gaussian process with covariance function :

$$\mathrm{cov}\{\mathbb{U}(s), \mathbb{U}(t)\} = min(s,t) - st, \qquad s,t \in [0,1].$$

Cumulative Sum test for $\mathcal{H}_0$, serially independent data II

Example 2: test à la [Csörgő et Horváth(1997)]

$$T_n^{\#} = \max_{k=1,\dots,n-1} \frac{k(n-k)}{n^{3/2}} \sup_{\mathbf{x}\in\mathbb{R}^d} |F_{1:k}(\mathbf{x}) - F_{k+1:n}(\mathbf{x})|$$
$$= \sup_{s\in[0,1]} \sup_{\mathbf{x}\in\mathbb{R}^d} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} \{\mathbf{1}(\mathbf{X}_i \leq \mathbf{x}) - F_{1:n}(\mathbf{x})\} \right|,$$

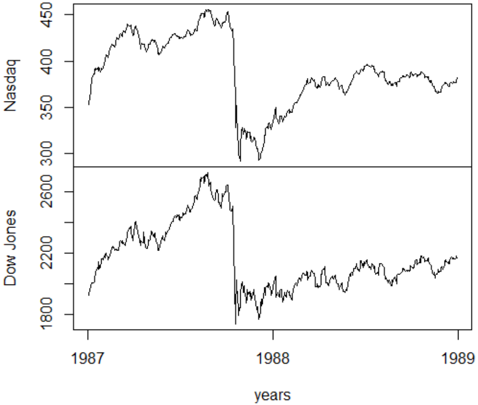
where for $1 \leq k \leq n$, $F_{1:k}$ (resp. $F_{k+1:n}$) is the empirical c.d.f. of the subsample $\mathbf{X}_1, \dots, \mathbf{X}_k$ (resp. $\mathbf{X}_{k+1}, \dots, \mathbf{X}_n$) :

$$F_{1:k}(\mathbf{x}) = \frac{1}{k} \sum_{i=1}^{k} \mathbf{1}(\mathbf{X}_i \leq \mathbf{x}), \qquad F_{k+1:n}(\mathbf{x}) = \frac{1}{n-k} \sum_{i=k+1}^{n} \mathbf{1}(\mathbf{X}_i \leq \mathbf{x}),$$

and

$$F_{1:n}(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(\mathbf{X}_i \leq \mathbf{x}).$$

Nasdaq, Dow Jones and the "black Monday" (1987-10-19)



▷: changes in the dependency between the two components?

Summary

The Sklar's theorem

Theorem of [Sklar(1959)]

Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a d-dimensional random vector with c.d.f. $\boldsymbol{F}$ and let $F_1, \ldots F_d$ be the marginal c.d.f. of $\boldsymbol{X}$ assuming continuous. Then it exists a unique function $C : [0,1]^d \to [0,1]$ such that:

$$\boldsymbol{F}(\boldsymbol{x}) = C\{F_1(x_1), \ldots, F_d(x_d)\}, \qquad \boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

- The copula C characterizes the dependence structure of vector $\boldsymbol{X}$.
- The copula C can be expressed as follows:

$$C(\boldsymbol{u}) = \boldsymbol{F}\{F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\}, \qquad \boldsymbol{u} = (u_1, \ldots, u_d) \in [0,1]^d.$$

A. Sklar.
Fonctions de répartition à n dimensions et leurs marges.
Publications de l'Institut de Statistique de l'Université de Paris, 8: 229–231, 1959.

Classical copulas

- Independence copula:

$$C^{\Pi}(\boldsymbol{u}) = \prod_{j=1}^{d} u_j;$$

- Normal copulas

$$C_{\Sigma}^{N}(\boldsymbol{u}) = \Phi_{d,\Sigma}\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\};$$

- Gumbel–Hougaard copulas:

$$C_{\theta}^{GH}(\boldsymbol{u}) = \exp\left(-\left[\sum_{j=1}^{d}\{-\log(u_j)\}^{\theta}\right]^{1/\theta}\right), \quad \theta \geq 1;$$

- Clayton copulas:

$$C_{\theta}^{Cl}(\boldsymbol{u}) = \left(\sum_{j=1}^{d} u_j^{-\theta} - d + 1\right)^{-1/\theta}, \quad \theta > 0.$$

→My Copulas here

The role of copulas to test for breaks detection

$$\mathcal{H}_0 : \exists\, F \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have c.d.f. } F.$$

Sklar's theorem allows to rewrite $\mathcal{H}_0$ as $\mathcal{H}_{0,m} \cap \mathcal{H}_{0,c}$ where

$\mathcal{H}_{0,m} \cap \mathcal{H}_{0,c}$:

$$\begin{aligned} \mathcal{H}_{0,c} : &\quad \exists C, \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have copula } C \\ \mathcal{H}_{0,m} : &\quad \exists F_1, \ldots, F_d \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have m.c.d.f. } F_1, \ldots, F_d\,. \end{aligned}$$

- Construction of a test for $\mathcal{H}_0$ more powerful than its predecessors against alternatives involving a change in the copula, based on the CUSUM approach.
- F, $F_1, \ldots, F_d$ and C are unknown.

Summary

Empirical copula I

- Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be d-dimensional random vectors with continuous m.c.d.f. $F_1, \ldots, F_d$ and copula C.
- For $i = 1, \ldots, n$, the random vectors $\boldsymbol{U}_i = (F_1(X_{i1}), \ldots, F_d(X_{id})) \sim C$.
- When $F_1, \ldots, F_d$ are supposed known, a natural estimator of C is given by the empirical c.d.f. of $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$.
- Here $F_1, \ldots, F_d$ are unknown, an estimator of C is given by the empirical c.d.f. of *pseudo-observations* of copula C:

For $j = 1, \ldots, d$ let $F_{1:n,j}$ be the empirical c.d.f. of sample $X_{1j}, \ldots, X_{nj}$. For $i = 1, \ldots, n$, consider the vectors

$$\hat{\boldsymbol{U}}_i^{1:n} = (F_{1:n,1}(X_{i1}), \ldots, F_{1:n,d}(X_{id})) = \frac{1}{n}(R_{i1}^{1:n}, \ldots, R_{id}^{1:n}),$$

where for $j = 1, \ldots, d$, $R_{ij}^{1:n}$ is the maximal rank of X_{ij} among $X_{1j}, \ldots, X_{nj}$.

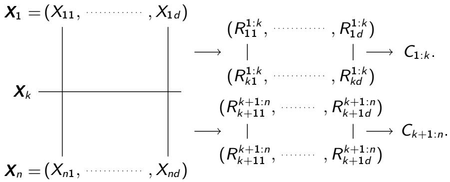
Empirical copula II

[Rüschendorf(1976)], [Deheuvels(1979)]

$$C_{1:n}(\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n} \leq \boldsymbol{u}), \qquad \boldsymbol{u} \in [0,1]^d.$$

Let $C_{1:k}$ (resp. $C_{k+1:n}$) the empirical copula evaluated on $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ (resp. $\boldsymbol{X}_{k+1}, \ldots, \boldsymbol{X}_n$):

$$C_{1:k}(\boldsymbol{u}) = \frac{1}{k}\sum_{i=1}^{k} \mathbf{1}(\hat{\boldsymbol{U}}_i^{1:k} \leq \boldsymbol{u}), \qquad C_{k+1:n}(\boldsymbol{u}) = \frac{1}{n-k}\sum_{i=k+1}^{n} \mathbf{1}(\hat{\boldsymbol{U}}_i^{k+1:n} \leq \boldsymbol{u}) \quad \boldsymbol{u} \in [0,1]^d.$$



Break detection in copula

We consider the process

$$\mathbb{D}_n(s, \boldsymbol{u}) = \sqrt{n}\frac{\lfloor ns \rfloor}{n}\frac{(n - \lfloor ns \rfloor)}{n}\{C_{1:\lfloor ns \rfloor}(\boldsymbol{u}) - C_{\lfloor ns \rfloor + 1:n}(\boldsymbol{u})\}, \qquad (s, \boldsymbol{u}) \in [0,1]^{d+1},$$

A Cramér-von Mises statistic:

$$S_{n,k} = \int_{[0,1]^d} \mathbb{D}_n^2(k/n, \boldsymbol{u}) \mathrm{d}C_{1:n}(\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{D}_n^2(k/n, \hat{\boldsymbol{U}}_i^{1:n}),$$

and

$$S_n = \max_{k \in \{1,\dots,n-1\}} S_{n,k}.$$

- Under $\mathcal{H}_0$, for all $k \in \{1, \dots, n-1\}$, $C_{1:k}$ and $C_{k+1:n}$ estimate the same copula C, thus S_n tends to be relatively weak.
- For an abrupt change in copula, the unknown break time can be estimate by the integer k which maximizes $S_{n,k}$.

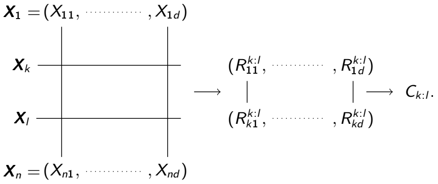
The sequential empirical copula process

Let for $s \le t \in [0,1]$, $\lambda_n(s,t) = (\lfloor nt \rfloor - \lfloor ns \rfloor)/n$.

Sequential empirical copula process

$$\begin{aligned}\mathbb{C}_n(s,t,\boldsymbol{u}) &= \sqrt{n}\lambda_n(s,t)\{C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - C(\boldsymbol{u})\} \\ &= \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left\{\mathbf{1}(\hat{\boldsymbol{U}}_i^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \le \boldsymbol{u}) - C(\boldsymbol{u})\right\}.\end{aligned}$$

$$C_{k:l}(\boldsymbol{u}) = \frac{1}{l-k+1}\sum_{i=k}^{l} \mathbf{1}(\hat{\boldsymbol{U}}_i^{k:l} \le \boldsymbol{u}) \quad \boldsymbol{u} \in [0,1]^d, \quad 1 \le k \le l \le n.$$



Asymptotic behaviour of the sequential empirical copula process

Condition: [Segers(2012)]

For any $j \in \{1, \ldots, d\}$, the partial derivatives $\dot{C}_j = \partial C / \partial u_j$ exist and are continuous on $V_j = \{\boldsymbol{u} \in [0,1]^d, u_j \in (0,1)\}$.

Theorem

Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be drawn from a strictly stationary sequence $(\boldsymbol{X}_i)_{i \in \mathbb{Z}}$ with continuous margins and whose the strong mixing coefficient satisfies $\alpha_r = O(r^{-a})$, $a > 1$. Under the previous condition,

$$\sup_{\boldsymbol{u} \in [0,1]^d} |\mathbb{C}_n(s,t,\boldsymbol{u}) - \widetilde{\mathbb{C}}_n(s,t,\boldsymbol{u})| \xrightarrow{\mathbb{P}} 0,$$

where for $\boldsymbol{u} \in [0,1]^d$, and $\boldsymbol{u}^{\{j\}} = (1, \ldots, 1, u_j, 1, \ldots, 1)$,

$$\widetilde{\mathbb{C}}_n(s,t,\boldsymbol{u}) = \mathbb{B}_n(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u}) \mathbb{B}_n(s,t,\boldsymbol{u}^{\{j\}}).$$

A decomposition for the process $\mathbb{D}_n$

$$\mathbb{B}_n(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \{1(\boldsymbol{U}_i \leq \boldsymbol{u}) - C(\boldsymbol{u})\}, \quad (s,t,\boldsymbol{u}) \in [0,1]^{d+2}, s \leq t.$$

$$\rightsquigarrow \mathbb{Z}_C(t,\boldsymbol{u}) - \mathbb{Z}_C(s,\boldsymbol{u})$$

in $\ell^\infty(\{s \leq t \in [0,1]^2\} \times [0,1]^d)$, $\mathbb{Z}_C$ a centered Gaussian process (a C-Kiefer–Müller process) [van der Vaart and Wellner(2000)]

The process $\mathbb{D}_n$ can be written as function of the sequential empirical copula process $\mathbb{C}_n$:

$$\mathbb{D}_n(s,\boldsymbol{u}) = \left(1 - \frac{\lfloor ns \rfloor}{n}\right) \mathbb{C}_n(0,s,\boldsymbol{u}) - \frac{\lfloor ns \rfloor}{n} \mathbb{C}_n(s,1,\boldsymbol{u}),$$

with $s \in [0,1]$ and $\boldsymbol{u} \in [0,1]^d$.

Asymptotic behaviour of S_n under $\mathcal{H}_0$

Proposition

Under $\mathcal{H}_0$

$$S_n = \sup_{s \in [0,1]} \int_{[0,1]^d} \{\mathbb{D}_n(s, \boldsymbol{u})\}^2 \mathrm{d}C_n(\boldsymbol{u}) \rightsquigarrow S_C = \sup_{s \in [0,1]} \int_{[0,1]^d} \{\mathbb{D}_C(s, \boldsymbol{u})\}^2 \mathrm{d}C(\boldsymbol{u}).$$

$$\mathbb{D}_C(s, \boldsymbol{u}) = (1-s)\mathbb{C}_C(0, s, \boldsymbol{u}) - s\mathbb{C}_C(s, 1, \boldsymbol{u}), \quad (s, \boldsymbol{u}) \in [0,1]^{d+1}.$$

and for $s \leq t \in [0,1]^2$ and $\boldsymbol{u} \in [0,1]^d$

$$\mathbb{C}_C(s, t, \boldsymbol{u}) = \{\mathbb{Z}_C(t, \boldsymbol{u}) - \mathbb{Z}_C(s, \boldsymbol{u})\} - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u})\{\mathbb{Z}_C(t, \boldsymbol{u}^{\{j\}}) - \mathbb{Z}_C(s, \boldsymbol{u}^{\{j\}})\},$$

with $\mathbb{Z}_C$ the centered Gaussian process and $\boldsymbol{u}^{\{j\}} = (1, \ldots, 1, u_j, 1, \ldots, 1)$.

Summary

A resampling scheme of the sequential empirical copula process

- From the previous theorem, the process $\mathbb{C}_n$ is asymptotically equivalent to process $\widetilde{\mathbb{C}}_n$

$$\widetilde{\mathbb{C}}_n(s,t,\boldsymbol{u}) = \{\mathbb{Z}_n(t,\boldsymbol{u}) - \mathbb{Z}_n(s,\boldsymbol{u})\} - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u})\{\mathbb{Z}_n(t,\boldsymbol{u}^{\{j\}}) - \mathbb{Z}_n(s,\boldsymbol{u}^{\{j\}})\},$$

with

$$\mathbb{Z}_n(s,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} \{\mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}) - C(\boldsymbol{u})\} \quad (s,\boldsymbol{u}) \in [0,1]^{d+1}.$$

- $\mathbb{Z}_n \rightsquigarrow \mathbb{Z}_C$ in $\ell^\infty([0,1]^{d+1})$, $\mathbb{Z}_C$ the C-Kiefer-Müller process
- To resample $\mathbb{C}_n$, we construct a resampling of $\mathbb{Z}_n$ and we estimate the partial derivatives $\dot{C}_j$.

A resampling scheme for $\mathbb{Z}_n$, case of i.i.d. observations

i.i.d. multipliers [van der Vaart and Wellner(2000)]

A sequence of i.i.d. multipliers $(\xi_i)_{i\in\mathbb{Z}}$ satisfies the following conditions:

- For all $i \in \mathbb{Z}$, ξ_i are independent of observations $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$
- $\mathbb{E}(\xi_0) = 0$, $\mathrm{var}(\xi_0) = 1$ and $\int_0^\infty \{\mathrm{P}(|\xi_0| > x)\}^{1/2} \mathrm{d}x < \infty$.

For $m = 1, \dots, M$ consider the processes

$$\mathbb{Z}_n^{(m)}(s, \boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_i^{(m)} \{\mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}) - C(\boldsymbol{u})\}, \qquad (s, \boldsymbol{u}) \in [0,1]^{d+1}.$$

[Holmes, Kojadinovic et Quessy(2013)]

$$\left(\mathbb{Z}_n, \mathbb{Z}_n^{(1)}, \dots, \mathbb{Z}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{Z}_C, \mathbb{Z}_C^{(1)}, \dots, \mathbb{Z}_C^{(M)}\right)$$

in $\{\ell^\infty([0,1]^{d+1})\}^{M+1}$, where $\mathbb{Z}_C$ is the C-Kiefer-Müller process and the processes $\mathbb{Z}_C^{(1)}, \dots, \mathbb{Z}_C^{(M)}$ are independent copies of $\mathbb{Z}_C$.

A resampling scheme for $\mathbb{C}_n$ I

- For serial dependent data, we have to construct dependent multipliers $\boldsymbol{\xi}_n^{(m)} \sim \mathcal{N}_n(\mathbf{0}, \Sigma)$, $\Sigma_{ij} = \varphi(\frac{i-j}{\ell_n})$ with φ symmetric, $\varphi(0) = 1$ and $\varphi(x) = 0$ for all $|x| > 1$ and $\ell_n = o(n)$ (i.e. ℓ_n dependence)
- For $m = 1, \ldots, M$, consider the estimated processes:

$$\hat{\mathbb{Z}}_n^{(m)}(s, \boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_{i,n}^{(m)} \{\mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n} \leq \boldsymbol{u}) - C_{1:n}(\boldsymbol{u})\}, \quad (s, \boldsymbol{u}) \in [0,1]^{d+1},$$

and

$$\hat{\mathbb{B}}_n^{(m)}(s, t, \boldsymbol{u}) = \hat{\mathbb{Z}}_n^{(m)}(t, \boldsymbol{u}) - \hat{\mathbb{Z}}_n^{(m)}(s, \boldsymbol{u}) \qquad (s, t, \boldsymbol{u}) \in [0,1]^{d+2}.$$

- We consider an estimator $\dot{C}_{j,1:n}$ of $\dot{C}_j$ constructed with finite differencing of the empirical copula at a bandwidth of h_n:

$$\dot{C}_j^{1:n}(\boldsymbol{u}) = \frac{C_{1:n}(\boldsymbol{u} + h_n \boldsymbol{e}_j) - C_{1:n}(\boldsymbol{u} - h_n \boldsymbol{e}_j)}{\min(u_j + h_n, 1) - \max(u_j - h_n, 0)}, \quad \boldsymbol{u} \in [0,1]^d,$$

$\boldsymbol{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ and $h_n = n^{-1/2}$.

A resampling scheme for $\mathbb{C}_n$ II

For the estimated processes

$$\hat{\mathbb{C}}_n^{(m)}(s,t,\boldsymbol{u}) = \hat{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_{j,1:n}(\boldsymbol{u})\, \hat{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}^{\{j\}}), \quad (s,t,\boldsymbol{u}) \in [0,1]^{d+2}, s \leq t,$$

we have the following result:

[Bücher, Kojadinovic, Rohmer et Segers(2014)]

Let $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ be drawn from a strictly stationary sequence $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$. Under "mixing conditions",

$$\left(\mathbb{C}_n, \hat{\mathbb{C}}_n^{(1)}, \dots, \hat{\mathbb{C}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{C}_C, \mathbb{C}_C^{(1)}, \dots, \mathbb{C}_C^{(M)}\right),$$

in $\{\ell^\infty(\Delta \times [0,1]^d)\}^{M+1}$, where $\mathbb{C}_C^{(1)}, \dots, \mathbb{C}_C^{(M)}$ are independent copies of $\mathbb{C}_C$.

A resampling scheme for $\mathbb{C}_n$ III

For $(s, t, \boldsymbol{u}) \in [0,1]^{d+2}$, $s \leq t$, consider the processes

$$\check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \xi_{i,n}^{(m)} \{\mathbf{1}(\hat{\boldsymbol{U}}_i^{\lfloor ns \rfloor+1:\lfloor nt \rfloor} \leq \boldsymbol{u}) - C_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u})\}.$$

and

$$\check{\mathbb{C}}_n^{(m)}(s,t,\boldsymbol{u}) = \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_{j,\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u})\, \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}^{\{j\}}).$$

[Bücher, Kojadinovic, Rohmer et Segers(2014)]

Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be drawn from a strictly stationary sequence $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$. Under the same mixing conditions:

$$\left(\mathbb{C}_n, \check{\mathbb{C}}_n^{(1)}, \ldots, \check{\mathbb{C}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{C}_C, \mathbb{C}_C^{(1)}, \ldots, \mathbb{C}_C^{(M)}\right),$$

in $\{\ell^\infty(\Delta \times [0,1]^d)\}^{M+1}$.

A resampling for S_n

- The process $\mathbb{D}_n$ can be rewritten as:

$$\mathbb{D}_n(s, \boldsymbol{u}) = \lambda_n(s, 1)\mathbb{C}_n(0, s, \boldsymbol{u}) - \lambda_n(0, s)\mathbb{C}_n(s, 1, \boldsymbol{u}).$$

- Two possibilities to resample $\mathbb{D}_n$:

$$\hat{\mathbb{D}}_n^{(m)}(s, \boldsymbol{u}) = \lambda_n(s, 1)\hat{\mathbb{C}}_n^{(m)}(0, s, \boldsymbol{u}) - \lambda_n(0, s)\hat{\mathbb{C}}_n^{(m)}(s, 1, \boldsymbol{u}),$$

$$\check{\mathbb{D}}_n^{(m)}(s, \boldsymbol{u}) = \lambda_n(s, 1)\check{\mathbb{C}}_n^{(m)}(0, s, \boldsymbol{u}) - \lambda_n(0, s)\check{\mathbb{C}}_n^{(m)}(s, 1, \boldsymbol{u}).$$

- ... and for S_n:

$$\hat{S}_n^{(m)} = \max_{k \in \{1, \dots, n-1\}} \sum_{i=1}^{n} \{\hat{\mathbb{D}}_n^{(m)}(k/n, \hat{\boldsymbol{U}}_i^{1:n})\}^2/n,$$

$$\check{S}_n^{(m)} = \max_{k \in \{1, \dots, n-1\}} \sum_{i=1}^{n} \{\check{\mathbb{D}}_n^{(m)}(k/n, \hat{\boldsymbol{U}}_i^{1:n})\}^2/n.$$

Asymptotic validity of the resampling scheme

[Bücher, Kojadinovic, Rohmer et Segers(2014)]

Under $\mathcal{H}_0$ and with the same mixing conditions and $\xi_{i,n}$ "well-chosen",

$$(S_n, \hat{S}_n^{(1)}, \ldots, \hat{S}_n^{(M)}) \rightsquigarrow (S_C, S_C^{(1)}, \ldots, S_C^{(M)})$$

$$(S_n, \check{S}_n^{(1)}, \ldots, \check{S}_n^{(M)}) \rightsquigarrow (S_C, S_C^{(1)}, \ldots, S_C^{(M)})$$

in $\mathbb{R}^{M+1}$, $S_C^{(1)}, \ldots, S_C^{(M)}$ independent copies of S_C.

- $\hat{S}_n^{(1)}, \ldots, \hat{S}_n^{(M)}$ and $\check{S}_n^{(1)}, \ldots, \check{S}_n^{(M)}$ can be interpreted as M 'almost' independent copies of S_n.
- We compute two approximate p-values for S_n as

$$\hat{p}_{M,n} = \frac{1}{M} \sum_{m=1}^{M} \mathbf{1}\left(\hat{S}_n^{(m)} \geq S_n\right) \text{ and } \check{p}_{M,n} = \frac{1}{M} \sum_{m=1}^{M} \mathbf{1}\left(\check{S}_n^{(m)} \geq S_n\right)$$

The pros and cons

+: Most powerful test than predecessors for detect a change in copula

+: Consistent tests

+: Adapted in the case of strong mixing observations

-: Expensive computation for big data analysis

-: Less sensitive to detect a change in marginal distribution

-: Without the hypothesis than the margins are constant; we can't conclude in favor of a change in copula.

Rohmer, T.
Non Parametric Test for Detecting Changes in the Copula
npCopTest, CRAN, 2018

Summary

Cramér-von Mises statistic, case of serially independent observations

Consider the alternative $\mathcal{H}_{1,c}$:

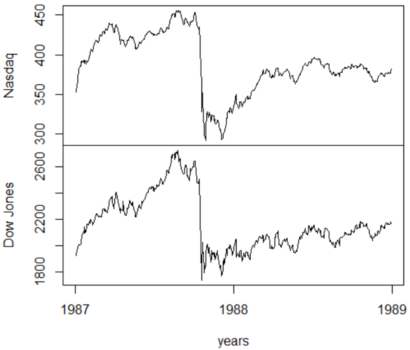
$\mathcal{H}_{1,c}$

$$H_{1,c} : \exists \text{ distinct } C_1 \text{ and } C_2, \text{ and } t \in (0,1) \text{ such that}$$
$$\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{\lfloor nt \rfloor} \text{ have copula } C_1 \text{ and } \boldsymbol{X}_{\lfloor nt \rfloor + 1}, \ldots, \boldsymbol{X}_n \text{ have copula } C_2.$$

Percentages of rejection of hypothesis $\mathcal{H}_0$ computed using 1000 samples of size $n \in \{50, 100, 200\}$ generated under:

- $\mathcal{H}_0 = \mathcal{H}_{0,c} \cap \mathcal{H}_{0,m}$,
- $\mathcal{H}_1 = \mathcal{H}_{1,c} \cap \mathcal{H}_{0,m}$

Nasdaq, Dow Jones and the "black Monday" (1987-10-19)



Suppose there is at more a unique change in m.c.d.f. at time $m = 202$ (1987-10-19)

▷ $\hat{p}_{val}^{S_{n,m}} = 0.201$: no evidence against $H_{0,c}$.

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Summary

Problematic

Let consider N the number of animals in our data set and $\underline{\boldsymbol{y}}_1 = (\boldsymbol{y}_{1,1}, \boldsymbol{y}_{1,2}), \ldots, \underline{\boldsymbol{y}}_N = (\boldsymbol{y}_{N,1}, \boldsymbol{y}_{N,2})$ the observed couples of phenotype. These phenotypes were measured n_i times, thus, for $i = 1, \ldots, N$,

$$\underline{\boldsymbol{y}}_i = (\boldsymbol{y}_{i,1}, \boldsymbol{y}_{i,2}) = \begin{bmatrix} y_{i11} & y_{i12} \\ \vdots & \vdots \\ y_{in_i1} & y_{in_i2} \end{bmatrix}.$$

1. For $i = 1, \ldots, N$, and $j = 1, 2$, $y_{i1j}, \ldots y_{in_ij}$ are potentially time-dependent
2. Phenotypes $\boldsymbol{y}_{i,1}$ and $\boldsymbol{y}_{i,2}$ don't are independent
3. The observations $\underline{\boldsymbol{y}}_1, \ldots, \underline{\boldsymbol{y}}_n$ don't are independent (genetic part)

Let consider $H_{i,t}$ the cumulative distribution of the random vector (Y_{it1}, Y_{it2}) and $F^1_{i,t}$ and $F^2_{i,t}$ the respective marginal distributions of Y_{it1} and Y_{it2} for $i = 1, \dots, N$ and $t = 1, \dots, n_i$, assumed continuous. According to the Sklar's theorem, it exists an unique function $C_{i,t}$ such that

$$H_{i,t}(y_1, y_2) = C_{i,t}(F^1_{i,t}(y_1), F^2_{i,t}(y_2)), \quad \text{for all } (y_1, y_2) \in \mathbb{R}^2.$$

The copula $C_{i,t}$ characterize the dependence structure of the vector (Y_{it1}, Y_{it2}).

$$C_{i,t} \leftarrow \overset{\begin{matrix} F^1_{i,t} & F^2_{i,t} \end{matrix}}{\begin{bmatrix} y_{i11} & y_{i12} \\ \vdots & \vdots \\ \vdots & \vdots \\ y_{in_i1} & y_{in_i2} \end{bmatrix}} \overset{\longleftrightarrow}{pedigree} \overset{\begin{matrix} F^1_{k,t} & F^2_{k,t} \end{matrix}}{\begin{bmatrix} y_{k11} & y_{k12} \\ \vdots & \vdots \\ \vdots & \vdots \\ y_{kn_k1} & y_{kn_k2} \end{bmatrix}} \rightarrow C_{k,t}$$

We will start by assuming conditions

A.1 Marginal distributions do not change with time:

$$F^j_{i,1} = \ldots = F^j_{i,n_i} \equiv F^j_i \text{ for all } j = 1, 2, \text{ and } i = 1, \ldots, N$$

A.2 Copulas do not change with time:

$$C_{i,1} = \ldots = C_{i,n_i} \equiv C_i \text{ for all } i = 1, \ldots, N$$

A.3 For all $i = 1, \ldots, N$, and $j = 1, 2$, $y_{i1j}, \ldots y_{in_ij}$ are time-independant

or A.3 For all $i = 1, \ldots, N$, and $j = 1, 2$, $y_{i1j}, \ldots y_{in_ij}$ are strong-mixing

A.4 Copulas $C_{i,t}$ belong to $\mathcal{F}_i = \{C_{i,\theta}, \theta \in \Theta\}$ where $C_{i,\theta}$ are parametric copulas and $\mathcal{F}_i$ depends on pedigree.

Some questions:

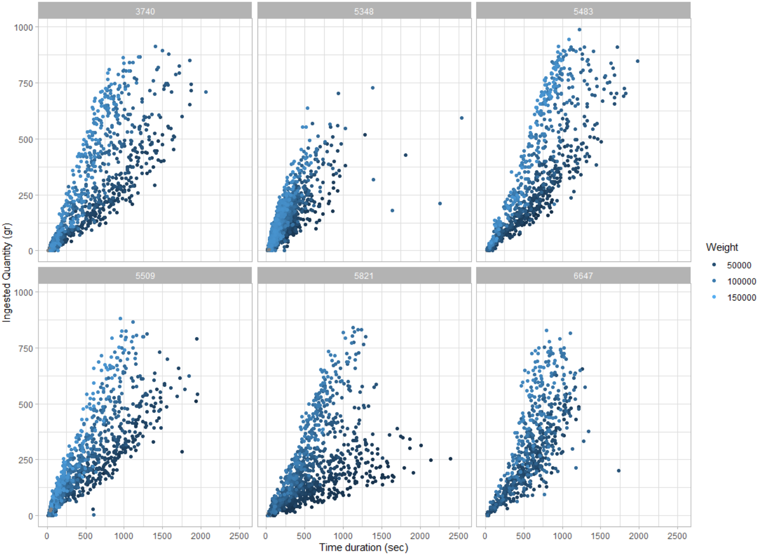
Q.1 Does dependence (copulas) depend on genetic effect?

Q.1.1 Does the genetic part affect the family of copula?

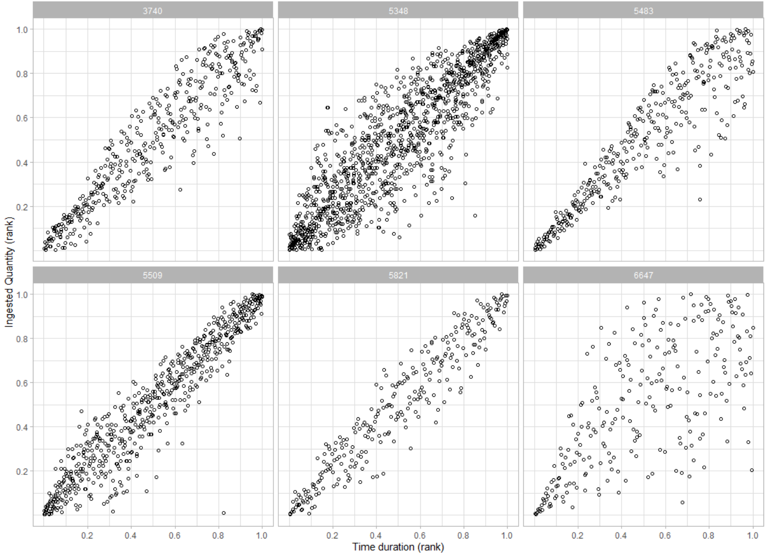
Q.1.2 Does the genetic part only affect parameters of copula?

Q.2 How to estimate the copula C_i for a specific individual taking into consideration the genetic effect?

Dac, Time durations & Ingested quantities



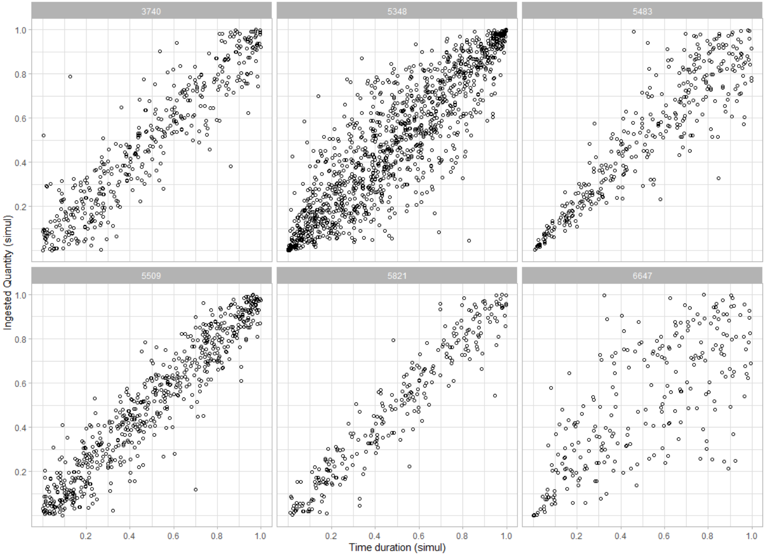
Dac, Ranked Time durations & Ranked Ingested quantities



Goodness-of-fit test

	n	ChangeTest	Copula	gofTest	tau
3740	422	0.669	Frank	0.55	0.77
5348	1235	0.486	t	0.08	0.70
5483	360	0.041	Clayton	0.17	0.72
5509	615	0.460	Frank	0.30	0.81
5821	271	0.197	Frank	0.02	0.81
6647	254	0.173	Clayton	0.23	0.52

Dac, Ranked Time durations & Ranked Ingested quantities (simulations)



Thank you for your attention!