Deux tests de détection de rupture dans la copule d'observations multivariées

par

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Résumé

Il est bien connu que les lois marginales d'un vecteur aléatoire ne suffisent pas à caractériser sa distribution. Lorsque les lois marginales du vecteur aléatoire sont continues, le théorème de Sklar garantit l'existence et l'unicité d'une fonction appelée *copule*, caractérisant la dépendance entre les composantes du vecteur. La loi du vecteur aléatoire est parfaitement définie par la donnée des lois marginales et de la copule.

Dans ce travail de thèse, nous proposons deux tests non paramétriques de détection de ruptures dans la distribution d'observations multivariées, particulièrement sensibles à des changements dans la copule des observations. Ils améliorent tous deux des propositions récentes et donnent lieu à des tests plus puissants que leurs prédécesseurs pour des classes d'alternatives pertinentes. Des simulations de Monte Carlo illustrent les performances de ces tests sur des échantillons de taille modérée.

Le premier test est fondé sur une statistique à la Cramér-von Mises construite à partir du *processus de copule empirique séquentiel*. Une procédure de rééchantillonnage à base de multiplicateurs est proposée pour la statistique de test ; sa validité asymptotique sous l'hypothèse nulle est démontrée sous des conditions de mélange fort sur les données.

Le second test se focalise sur la détection d'un changement dans le *rho de Spearman* multivarié des observations. Bien que moins général, il présente de meilleurs résultats en terme de puissance que le premier test pour les alternatives caractérisées par un changement dans le rho de Spearman. Deux stratégies de calcul de la valeur p sont comparées théoriquement et empiriquement : l'une utilise un rééchantillonnage de la statistique, l'autre est fondée sur une estimation de la loi limite de la statistique de test.

Mots-clés : tests non-paramétriques, données fortement mélangeantes, processus de copule empirique séquentiel, rééchantillonnage à base de multiplicateurs, rho de Spearman multidimensionnel, simulations de Monte Carlo.

Abstract

It is very well-known that the marginal distributions of a random vector do not characterize the distribution of the random vector. When the marginal distributions are continuous, the work of Sklar ensures the existence and uniqueness of a function called *copula* which can be regarded as capturing the dependence between the components of the random vector. The cumulative distribution function of the vector can then be rewritten using only the copula and the marginal cumulative distribution functions.

In this work, we propose two non-parametric tests for change-point detection, particularly sensitive to changes in the copula of multivariate time series. They improve on recent propositions and are more powerful for relevant alternatives involving a change in the copula. The finite-sample behavior of these tests is investigated through Monte Carlo experiments.

The first test is based on a Cramér-von Mises statistic and on the sequential empirical copula process. A multiplier resampling scheme is suggested and its asymptotic validity under the null hypothesis is demonstrated under strong mixing conditions.

The second test focuses on the detection of a change in *Spearman's rho*. Monte Carlo simulations reveal that this test is more powerful than the first test for alternatives characterized by a change in Spearman's rho. Two approaches to compute approximate p-values for the test are studied empirically and theoretically. The first one is based on resampling, the thecond one consists of estimating the asymptotic null distribution of the test statictic.

Keywords : non-parametric tests, strong mixing conditions, sequential empirical copula, multipliers bootstrap, multidimensional Spearman's rho, Monte Carlo experiments.

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Introduction

Dans la modélisation statistique classique, l'hypothèse principale faite sur les données est que ces dernières sont issues d'une même loi de probabilité. Lorsque l'on dispose de données observées, une question très souvent d'intérêt est alors de chercher à savoir si les observations satisfont cette hypothèse. Malheureusement, cette hypothèse n'est pas toujours réaliste. À partir d'une certaine observation, la loi de ces dernières peut changer. Le changement peut se faire de façon abrupte ou graduelle.

Les procédures statistiques permettant de tester la présence d'un ou plusieurs changements dans la loi des observations sont appelées *tests de détection de rupture*. L'objet de ce travail de thèse est la mise en place de tests de détection de rupture *non paramétriques* pour des données *multivariées* et potentiellement *sériellement dépendantes*. De tels tests trouvent leurs applications dans de nombreux domaines, comme l'hydrologie, la climatologie, la fiabilité, ou encore la finance.

Il est bien connu que la fonction de répartition (f.d.r.) d'un vecteur aléatoire d-dimensionnel $\boldsymbol{X} = (X_1, \dots, X_d)$, définie pour $(x_1, \dots, x_d) \in \mathbb{R}^d$ par $F(x_1, \dots, x_d) = \mathrm{P}(X_1 \leq x_1, \dots, X_d \leq x_d)$, caractérise la loi du vecteur. C'est pourquoi nous allons nous intéresser à construire des tests statistiques non paramétriques, particulièrement sensibles à un changement dans la f.d.r. d'observations multivariées. Notons $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ les vecteurs aléatoires constituant nos observations. L'hypothèse $\mathcal{H}_0$ que l'on souhaite tester est la suivante :

$$\mathcal{H}_0 : \quad \text{Il existe } F \text{ telle que } F \text{ est la f.d.r. des vecteurs aléatoires } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n,$$

où la f.d.r. F est inconnue.

Dans le cas de données univariées et indépendantes, Pettitt (1979) considère une approche dite *cumulative sum* (CUSUM) et des statistiques de type Kolmogorov-Smirnov pour la mise en place de tests non paramétriques de détection de rupture. Des tests plus sensibles à des ruptures multiples et des changements graduels sont proposés dans Lombard (1987). L'approche de Pettitt est également adaptée au cas de données multivariées dans Csörgő et Horváth (1997), Gombay et Horváth (1999) ainsi que dans Holmes et coll. (2013). Le cas de données multivariées et sériellement dépendantes est traité dans Inoue (2001). Dans Gombay et Horváth (1999) en particulier, on pourra trouver des procédures basées sur un rééchantillonnage de la statistique de test, menant à des tests consistants. Cette approche est revisitée et généralisée dans Holmes et coll. (2013).

Une alternative possible à l'hypothèse $\mathcal{H}_0$ considérée précédemment est le cas d'une rupture dans au moins une des f.d.r. marginales des vecteurs aléatoires $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ d-dimensionnels. En d'autres termes, une alternative à l'hypothèse $\mathcal{H}_0$ peut être l'hypothèse $\neg(\mathcal{H}_{0,m})$, où $\mathcal{H}_{0,m}$ est définie par :

$$\mathcal{H}_{0,m}: \quad \text{Il existe } F_1, \dots, F_d \text{ telles que } F_1, \dots, F_d \text{ sont les f.d.r. marginales des vecteurs aléatoires } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n.$$

Néanmoins, les f.d.r. marginales d'un vecteur aléatoire ne caractérisent pas la loi du vecteur aléatoire en question. Ainsi l'hypothèse $\mathcal{H}_{0,m}$ n'implique pas l'hypothèse $\mathcal{H}_0$. Lorsque les f.d.r. marginales sont continues, ce constat peut être mis en évidence à l'aide du théorème principal apparaissant dans Sklar (1959). Celui-ci permet de relier la f.d.r. d'un vecteur aléatoire aux f.d.r. marginales au travers d'une fonction appelée «copule». Plus particulièrement, si on considère un vecteur aléatoire $\boldsymbol{X}$ de dimension d, de f.d.r F et dont les f.d.r. marginales $F_1, \dots, F_d$ sont continues, le théorème de Sklar dit qu'il existe une unique fonction $C : [0,1]^d \to [0,1]$ appelée «copule du vecteur $\boldsymbol{X}$» telle que

$$F(x_1, \dots, x_d) = C\{F_1(x_1), \dots, F_d(x_d)\} \qquad \boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

La copule caractérise la dépendance entre les composantes du vecteur aléatoire $\boldsymbol{X}$ (d'où son appellation de «fonction de dépendance», que l'on retrouve dans Deheuvels, 1979). On peut alors voir que l'hypothèse $\mathcal{H}_0$ peut s'écrire comme $\mathcal{H}_{0,m} \cap \mathcal{H}_{0,c}$ où $\mathcal{H}_{0,c}$ est définie par :

$$\mathcal{H}_{0,c}: \quad \text{Il existe } C \text{ telle que } C \text{ est la copule des vecteurs aléatoires } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n.$$

Bien que consistant pour l'alternative générale de rupture dans la f.d.r. des observations, les simulations de Monte Carlo que l'on trouve Holmes et coll. (2013), révèlent que les tests non paramétriques multivariés cités précédemment, semblent peu puissants pour des échantillons de taille modérée, pour toute hypothèse alternative de la forme $\mathcal{H}_{0,m} \cap \neg(\mathcal{H}_{0,c})$.

La première contribution de la thèse est la proposition d'un test non paramétrique de détection de rupture particulièrement sensible à l'alternative

$\mathcal{H}_{0,m} \cap \neg(\mathcal{H}_{0,c})$ et adaptée au cas de données sériellement dépendantes. De façon similaire à l'approche de Rémillard et Scaillet (2009), reprise dans Bücher et Ruppert (2013) et dans Bücher et Kojadinovic (2013), un rééchantillonnage à base de multiplicateurs de la statistique est employé. L'étude asymptotique sous $\mathcal{H}_0$ de ce dernier nécessite une condition de régularité portant sur les dérivées partielles de la copule.

Dans certains domaines comme en finance, on cherche de façon plus spécifique à détecter un changement dans la corrélation croisée d'observations issues de séries temporelles multivariées. La deuxième contribution de cette thèse est la proposition d'un test non paramétrique de détection de rupture particulièrement sensible à un changement dans le *rho de Spearman multivarié*. Un tel test devrait alors être plus puissant pour $\mathcal{H}_{0,m} \cap \neg(\mathcal{H}_{0,c})$ sous l'hypothèse supplémentaire d'une rupture dans le rho de Spearman multivarié. Des tests similaires fondés sur le *tau de Kendall* ont été proposés par Gombay et Horváth (1999, 2002), Quessy et coll. (2013) et Dehling et coll. (2014).

Notre test améliore la proposition similaire de Wied et coll. (2014) dans le sens où, contrairement à la statistique que nous utilisons, celle des auteurs cités précédemment ne peut pas s'écrire comme différence pondérée de rho de Spearman empiriques. L'étude du comportement asymptotique de la statistique considérée dans cette seconde contribution ne nécessite plus de conditions portant sur les dérivées partielles de la copule. La mise en œuvre du test peut être effectuée en utilisant un rééchantillonnage de la statistique à base de multiplicateurs ou en estimant la loi limite de la statistique sous l'hypothèse $\mathcal{H}_0$.

Dans la première partie de cette thèse, nous présentons de façon plus précise l'outil «copule» et nous faisons un état de l'art sur la copule empirique, le processus de copule empirique séquentiel ainsi que le rééchantillonnage associé, pour des données indépendantes puis pour des données sériellement dépendantes. La seconde partie concerne l'ensemble des travaux menés et sera présentée sous la forme d'articles. Ces articles portent sur la mise en place de tests statistiques pour la détection de rupture dans la copule mentionné ci-dessus. La validité asymptotique des tests sous $\mathcal{H}_0$, est établie de façon rigoureuse et leurs comportements pour des échantillons de petites tailles sont illustrés par des simulations de Monte Carlo. Le chapitre 3 contient l'article «Detecting changes in cross-sectional dependence in multivariate time series» co-écrit avec Axel Bücher, Ivan Kojadinovic et Johan Segers. Le dernier chapitre expose l'article «Testing the constancy of Spearman's rho in multivariate time series» co-écrit avec Ivan Kojadinovic et Jean-François Quessy.

Première partie

Outils probabilistes de mesure de la dépendance et versions empiriques associées

Chapitre 1

Modélisation de la dépendance multivariée à l'aide des copules

L'étude des copules est un phénomène relativement récent et en plein essor. Ces dernières sont utilisées dans de nombreuses applications, notamment dans le domaine de l'hydrologie, de la finance et de la gestion du risque. Dans ces domaines, on cherche très souvent à modéliser la dépendance entre les composantes de chaque vecteur formant les données. La notion de copule apparaît la première fois sous le terme *copula* dans l'article de Sklar (1959), comme étant la fonction qui va «combiner» les f.d.r. marginales du vecteur aléatoire pour former la f.d.r. du vecteur. Ce chapitre a pour but de présenter l'outil «copule» et ses principales caractéristiques, ainsi que ses liens avec d'autres mesures de dépendance classiques comme le rho de Spearman et le tau de Kendall. Pour une revue plus exhaustive sur la notion de copule, le lecteur pourra consulter le livre de Nelsen (2006) ainsi que le chapitre 5 de McNeil et coll. (2005) ou encore Durante et Sempi (2010).

Sommaire

1.1 Généralités sur la notion de copule

Dans ce document de thèse, afin de faciliter la lecture, les vecteurs de $\mathbb{R}^d$ seront notés en gras par des lettres minuscules, les variables aléatoires de $\mathbb{R}$ seront notées par des lettres majuscules et enfin, les vecteurs aléatoires de $\mathbb{R}^d$ seront notés en gras par des lettres majuscules.

Soit $d \geq 1$ un entier positif. Pour deux vecteurs $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, la fonction $(\boldsymbol{x}, \boldsymbol{y}) \mapsto \mathbf{1}(\boldsymbol{x} \leq \boldsymbol{y})$ (resp. $\mathbf{1}\{\boldsymbol{x} = \boldsymbol{y}\}$) est définie comme étant la fonction indicatrice valant 1 si $\boldsymbol{x} \leq \boldsymbol{y}$ (resp. si $\boldsymbol{x} = \boldsymbol{y}$) et sinon 0. Notons que les inégalités (resp. égalités) entre deux vecteurs de $\mathbb{R}^d$ s'interprètent ici (et dans la suite de ce document) composantes par composantes, c'est-à-dire, $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$,

$$\boldsymbol{x} \leq \boldsymbol{y} \Leftrightarrow x_j \leq y_j \quad \forall j \in \{1, \ldots, d\}.$$

Définition 1.1.1. *Une copule C de dimension $d \geq 2$ est la fonction de répartition sur $[0,1]^d$ d'un vecteur aléatoire $\boldsymbol{U}$ de dimension d dont les lois marginales sont uniformes sur $[0,1]$. En d'autres termes,*

$$C(\boldsymbol{u}) = P(\boldsymbol{U} \leq \boldsymbol{u}), \qquad \boldsymbol{u} \in [0,1]^d.$$

Pour un sous-ensemble A de $\{1, \dots, d\}$ et un vecteur $\boldsymbol{u}$ de $[0,1]^d$, nous définissons le vecteur $\boldsymbol{u}^A$ comme étant le vecteur de $[0,1]^d$ dont les composantes sont données par :

$$u_j^A = \begin{cases} u_j, & \text{si } j \in A, \\ 1, & \text{sinon.} \end{cases} \tag{1.1}$$

De la définition 1.1.1 découle la propriété suivante :

Propriété 1.1.1. *Une fonction $C : [0,1]^d \to [0,1]$ est une copule si et seulement si :*

1. *pour tout $\boldsymbol{u}, \boldsymbol{v} \in [0,1]^d$ tels que $\boldsymbol{u} \leq \boldsymbol{v}$, $C(\boldsymbol{u}) \leq C(\boldsymbol{v})$;*
2. *pour tout $j \in \{1, \dots, d\}$, $C(\boldsymbol{u}^{\{j\}}) = u_j$;*
3. *pour tous couples $(a_1, b_1), \dots, (a_d, b_d) \in [0,1]^2$ tels que $a_j \leq b_j$ pour tout $j \in \{1, \dots, d\}$,*

$$\sum_{(u_1,\dots,u_d) \in \{a_1,b_1\} \times \dots \times \{a_d,b_d\}} (-1)^{\sum_{j=1}^d \mathbf{1}(u_j = a_j)} C(u_1, \dots, u_d) \geq 0.$$

La condition 1 est la condition d'isotonicité. Elle est nécessaire pour garantir l'appartenance de la copule C à l'ensemble des f.d.r. sur $\mathbb{R}^d$. La condition 2 garantit que les lois marginales du vecteur aléatoire $\boldsymbol{U}$ sont bien uniformes sur [0,1]. Enfin, la condition 3 est la condition de d-croissance (croissance d-dimensionnelle). On peut la réécrire dans le cas $d = 2$ comme suit :

$$C(a_1, a_2) - C(a_1, b_2) - C(b_1, a_2) + C(b_1, b_2) \geq 0,$$

pour tous $a_1, a_2, b_1, b_2 \in [0,1]$ tels que $a_1 \leq b_1, a_2 \leq b_2$. La somme apparaissant dans la condition 3 est généralement appelée volume de C sur $[\boldsymbol{a}, \boldsymbol{b}] = [a_1, b_1] \times \dots \times [a_d, b_d]$. En fait, cette dernière condition assure que si le vecteur aléatoire $\boldsymbol{U}$ admet C pour fonction de répartition sur $[0,1]^d$, alors la probabilité que $\boldsymbol{U}$ appartienne à n'importe quel «hyperpavé» $[\boldsymbol{a}, \boldsymbol{b}]$ de $[0,1]^d$ est positive ou nulle. En d'autres termes :

$$\mathrm{P}(a_1 \leq U_1 \leq b_1, \dots, a_d \leq U_d \leq b_d) \geq 0,$$

pour tous $a_1 \leq b_1, \dots, a_d \leq b_d$.

La fonction C étant une f.d.r. sur $[0,1]^d$, elle est nulle sur les bords gauches de son domaine, c'est-à-dire :

$$C(\boldsymbol{u}) = 0 \text{ pour tout } \boldsymbol{u} \in [0,1]^d \text{ tel que } \exists j \in \{1,\ldots,d\} \text{ pour lequel } u_j = 0.$$

Enfin, C est une fonction Lipschitzienne, c'est-à-dire qu'elle vérifie la propriété suivante :

$$\forall \boldsymbol{u}, \boldsymbol{v} \in [0,1]^d, \quad |C(\boldsymbol{u}) - C(\boldsymbol{v})| \leq \sum_{j=1}^{d} |u_j - v_j|. \tag{1.2}$$

Étant données d f.d.r. univariées $F_1, \ldots, F_d$, il est possible de construire une infinité de f.d.r. sur $\mathbb{R}^d$ dont les marges sont $F_1, \ldots, F_d$. Le théorème de Sklar (1959) met en lumière ce fait. Il permet de relier la f.d.r. d'un vecteur aléatoire de dimension d à ses f.d.r. marginales au travers d'une copule sur $[0,1]^d$. En outre, il permet la construction de copules sur $[0,1]^d$ à partir de n'importe quelle f.d.r. sur $\mathbb{R}^d$. De façon réciproque, il amène à la construction de f.d.r. sur $\mathbb{R}^d$ à partir de d f.d.r. univariées quelconques et de n'importe quelle copule sur $[0,1]^d$. Le théorème est le suivant :

Théorème 1.1.1. *Soit F une fonction de répartition sur $\mathbb{R}^d$ de marges $F_1, \ldots, F_d$. Alors il existe une copule $C : [0,1]^d \to [0,1]$ telle que,*

$$F(\boldsymbol{x}) = C\{F_1(x_1), \ldots, F_d(x_d)\}, \quad \text{pour tout } \boldsymbol{x} \in \bar{\mathbb{R}}^d, \tag{1.3}$$

où $\bar{\mathbb{R}} = [-\infty, +\infty]$. Si, de plus, les marges $F_1, \ldots, F_d$ sont continues, alors C est déterminée de façon unique. Lorsque les marges ne sont pas continues, la copule C est déterminée de façon unique seulement sur $F_1(\bar{\mathbb{R}}) \times \ldots \times F_d(\bar{\mathbb{R}})$.

Réciproquement, si C est une copule et $F_1, \ldots, F_d$ sont des f.d.r. sur $\mathbb{R}$, alors la fonction F définie dans (1.3) est une f.d.r. sur $\mathbb{R}^d$ dont les marges sont $F_1, \ldots, F_d$.

Dans la suite de ce travail, nous nous placerons uniquement dans le cas où les f.d.r. marginales $F_1, \ldots, F_d$ de $\boldsymbol{X}$ sont continues, ceci afin de garantir l'unicité de C sur $[0,1]^d$. L'unique copule décrite dans la première partie du théorème 1.3 est alors donnée par

$$C(\boldsymbol{u}) = F(\boldsymbol{F}^{-1}(\boldsymbol{u})), \qquad \boldsymbol{u} \in [0,1]^d, \tag{1.4}$$

où

$$\boldsymbol{F}^{-1}(\boldsymbol{u}) = (F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)), \qquad F_j^{-1}(u_j) = \inf\{v \in [0,1] : F_j(v) \geq u_j\},$$

pour tout $j \in \{1, \ldots, d\}$ et $\boldsymbol{u} \in [0,1]^d$. L'expression (1.4) est obtenue naturellement en évaluant F définie dans (1.3) au point $F^{-1}(\boldsymbol{u})$.

Définition 1.1.2. *Soit $\boldsymbol{X}$ un vecteur aléatoire de dimension d de f.d.r. F et de f.d.r. marginales $F_1, \ldots, F_d$ continues. On appellera copule de $\boldsymbol{X}$ la f.d.r. C du vecteur aléatoire $(F_1(X_1), \ldots, F_d(X_d))$.*

La copule C d'un vecteur aléatoire d-dimensionnel $\boldsymbol{X}$ permet de modéliser la dépendance entre les composantes du vecteur $\boldsymbol{X}$. Une autre propriété importante pour la copule C est la propriété d'invariance suivante :

Proposition 1.1.1. *Soit $\boldsymbol{X}$ un vecteur aléatoire de dimension d, de marges $F_1, \ldots, F_d$ continues et de copule C. Considérons $t_1, \ldots, t_d$, d applications réelles, strictement croissantes. Alors le vecteur $(t_1(X_1), \ldots, t_d(X_d))$ admet C pour copule.*

Enfin, une dernière propriété à noter concerne les bornes de Fréchet-Hoeffding des copules. Considérons les fonctions W et M allant de $[0,1]^d$ dans $\mathbb{R}$, définies par

$$W(\boldsymbol{u}) = \max\{u_1 + \ldots + u_d - (d-1), 0\}, \qquad M(\boldsymbol{u}) = \min(u_1, \ldots, u_d), \quad \boldsymbol{u} \in [0,1]^d. \tag{1.5}$$

Proposition 1.1.2. *Pour tout $\boldsymbol{u} \in [0,1]^d$, on a*

$$W(\boldsymbol{u}) \leq C(\boldsymbol{u}) \leq M(\boldsymbol{u}), \qquad \boldsymbol{u} \in [0,1]^d.$$

La fonction M est la copule dite de *comonotonicité*. C'est en fait la f.d.r. du vecteur aléatoire d-dimensionnel $(U, \ldots, U)$ où U suit la loi uniforme sur $[0,1]$. La fonction W quant à elle est une copule au sens de la définition 1.1.1 uniquement lorsque $d = 2$. On l'appelle fréquemment copule d'*antimonotonicité*. Il s'agit de la f.d.r. du vecteur aléatoire $(U, 1-U)$ où U suit la loi uniforme sur $[0,1]$. On pourra vérifier que pour $d > 2$, W ne satisfait pas la condition 3 de la proposition 1.1.1 sur l'hypercube $[1/2, 1]^d$. Les détails du calcul peuvent être trouvés dans l'exemple 5.21 de McNeil et coll. (2005).

À titre d'exemple, nous présentons d'autres copules classiques. La copule la plus évidente est sans doute la copule d'indépendance définie par

$$\Pi(\boldsymbol{u}) = \prod_{j=1}^{d} u_j, \qquad \boldsymbol{u} = (u_1, \ldots, u_d) \in [0,1]^d. \tag{1.6}$$

On pourra remarquer que les composantes d'un vecteur aléatoire $\boldsymbol{X}$ sont indépendantes si et seulement si Π est la copule du vecteur $\boldsymbol{X}$.

Parmi les familles paramétriques de copules les plus fréquemment utilisées, on trouve celles de Gumbel–Hougaard, de Clayton, de Frank ou encore la famille des copules normales et de Student. Nous les utiliserons dans les simulations de la partie 2. Leurs expressions en dimension d, en fonction d'un paramètre réel θ, sont respectivement données pour $\boldsymbol{u} \in [0,1]^d$ par

$$C_\theta^{GH}(\boldsymbol{u}) = \exp\left(-\sum_{j=1}^{d}\left[\{-\log(u_j)\}^\theta\right]^{1/\theta}\right), \quad \theta \geq 1,$$
$$C_\theta^{Cl}(\boldsymbol{u}) = \max\left(\sum_{j=1}^{d} u_j^{-\theta} - d + 1, 0\right)^{-1/\theta}, \quad \theta > 0,$$
$$C_\theta^{F}(\boldsymbol{u}) = \frac{1}{\theta}\log\left[1 + \frac{\prod_{j=1}^{d}\{\exp(-\theta u_j) - 1\}}{\{\exp(-\theta) - 1\}^{d-1}}\right], \quad \theta \in \mathbb{R}^*.$$

L'interprétation du paramètre θ dépend de la famille considérée. Pour les copules de Gumbel–Hougaard lorsque θ vaut 1, on retrouve la copule d'indépendance, alors que lorsqu'il tend vers l'infini, on retrouve la copule de comonotonicité. En ce qui concerne les copules de Clayton, notons tout d'abord que, dans le cas bivarié, le domaine de θ peut être élargi à $[-1, 0[\cup\mathbb{R}_+^*$, le cas $\theta = -1$ correspondant à la copule W. Lorsque le paramètre θ tend vers 0, on retrouve la copule d'indépendance, et lorsque θ tend vers l'infini, on retrouve la copule de comonoticité. Des échantillons tirés de ces trois familles de copules sont représentés dans les figures 1.1 à 1.3, à l'aide du logiciel libre R statistical system (R Development Core Team, 2013), en utilisant le package `copula`.

Les copule normales, et de Student à $\nu > 0$ degrés de libertés (d.d.l.) sont quant à elles définies respectivement par,

$$C_\Sigma^N(\boldsymbol{u}) = \Phi_{d,\Sigma}\{\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\}, \qquad \boldsymbol{u} \in [0,1]^d$$
$$C_\Sigma^{t,\nu}(\boldsymbol{u}) = t_{d,\Sigma,\nu}\{t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)\},$$

où $\Phi_{d,\Sigma}$ (resp. $t_{d,\Sigma,\nu}$) est la fonction de répartition de la loi normale centrée réduite (resp. de student à ν d.d.l.) de dimension d, et de matrice de corrélation Σ, et Φ^{-1} (resp. t_ν^{-1}) est l'inverse de la f.d.r. de la loi normale centrée réduite unidimensionnelle (resp. de Student unidimensionnelle à ν d.d.l.). En dimension 2, on pourra retrouver d'autres familles paramétriques de copules dans la table 4.1 de Nelsen (2006).

1.2 Mesures de dépendance

Dans cette section, nous allons nous intéresser à deux mesures de dépendance particulièrement étudiées dans la littérature, à savoir le rho de Spearman et le tau

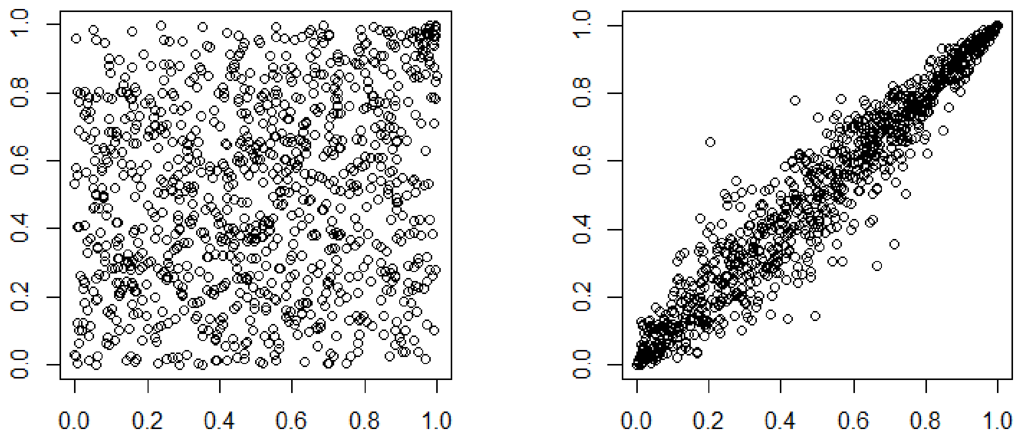

Figure 1.1 – Gauche (resp. droite) : 1000 réalisations d'un vecteur aléatoire (U_1, U_2) ayant pour copule la copule de Gumbel–Hougaard de paramètre $\theta = 1.2$ (resp. $\theta = 6$).

de Kendall, toutes deux fortement liées à la notion de copule dans le cas où les f.d.r. marginales sont continues. En effet, ces deux mesures peuvent alors s'écrire comme fonctionnelles de la copule sous-jacente à un vecteur aléatoire.

1.2.1 Rho de Spearman

Considérons le vecteur aléatoire bivarié $\boldsymbol{X} = (X_1, X_2)$. Lorsque les variances de X_1 et de X_2 sont finies, la mesure de dépendance la plus fréquemment employée est le coefficient de corrélation linéaire de Pearson entre X_1 et X_2. Ce dernier est défini par :

$$\operatorname{corr}(X_1, X_2) = \frac{\operatorname{cov}(X_1, X_2)}{\sqrt{\operatorname{Var}(X_1)\operatorname{Var}(X_2)}}.$$

Cette quantité varie entre -1 et 1, valant 1 (resp. -1) en cas de dépendance linéaire parfaite positive (resp. négative). Lorsque X_1 et X_2 sont indépendantes, la corrélation est nulle. En revanche la non-corrélation entre X_1 et X_2 n'implique bien entendu pas l'indépendance. On a la propriété d'invariance suivante :

Proposition 1.2.1. *Soient X_1 et X_2 deux variables aléatoires de variances finies. Considérons t_1 et t_2 deux applications linéaires strictement croissantes. Alors on*

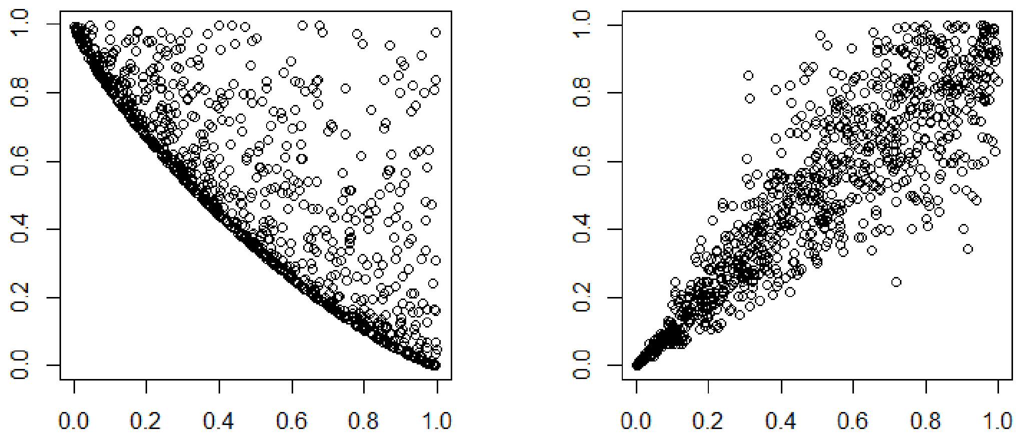

Figure 1.2 – Gauche (resp. droite) : 1000 réalisations d'un vecteur aléatoire (U_1, U_2) ayant pour copule la copule de Clayton de paramètre $\theta = -0.8$ (resp. $\theta = 6$).

peut écrire $\operatorname{corr}(t_1(X_1), t_2(X_2)) = \operatorname{corr}(X_1, X_2)$.

Notons F_1 et F_2 les f.d.r. respectives de X_1 et X_2 supposées continues. Le rho de Spearman du vecteur aléatoire (X_1, X_2), noté $\rho(X_1, X_2)$, est alors défini comme le coefficient de corrélation linéaire de Pearson du vecteur $\boldsymbol{U} = (F_1(X_1), F_2(X_2))$, c'est-à-dire,

$$\rho(X_1, X_2) = \frac{\mathbb{E}(U_1 U_2) - \mathbb{E}(U_1)\mathbb{E}(U_2)}{\sqrt{\operatorname{Var}(U_1)\operatorname{Var}(U_2)}}. \tag{1.7}$$

Notons que contrairement au coefficient de corrélation linéaire de Pearson, le rho de Spearman est toujours bien défini. Dans les cas où les f.d.r. marginales sont continues, il s'exprime en fonction de la copule C du vecteur $\boldsymbol{X}$ comme suit :

Proposition 1.2.2. *Soient X_1 et X_2 deux variables aléatoires de f.d.r. continue. Alors*

$$\rho(X_1, X_2) = \frac{\int_{[0,1]^2} uv \, \mathrm{d}C(u, v) - (1/2)^2}{\sqrt{1/12}\sqrt{1/12}} \tag{1.8}$$

$$= 12 \int_{[0,1]^2} C(u, v) \, \mathrm{d}u\mathrm{d}v - 3. \tag{1.9}$$

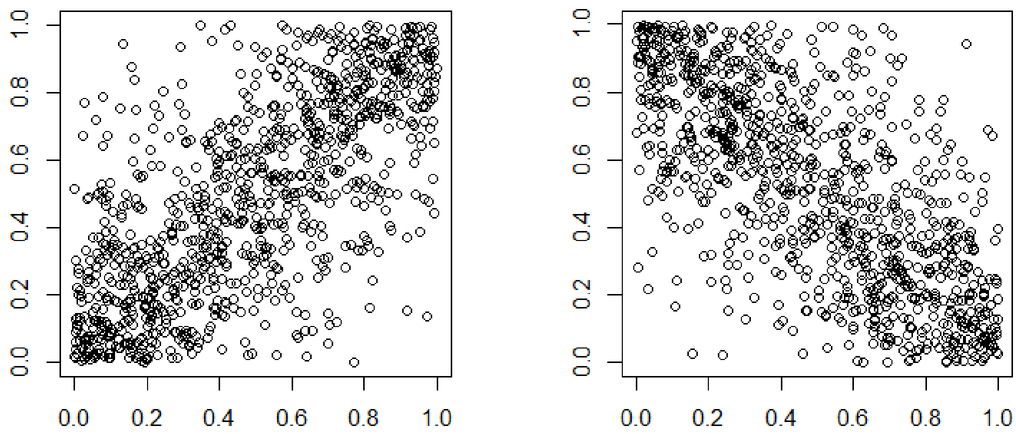

FIGURE 1.3 – Gauche (resp. droite) : 1000 réalisations d'un vecteur aléatoire (U_1, U_2) ayant pour copule la copule de Frank de paramètre $\theta = 6$ (resp. $\theta = -6$).

Démonstration. L'égalité (1.8) découle du fait que U_1 et U_2 sont uniformes sur $[0,1]$ et que $\boldsymbol{U}$ admet C pour f.d.r 2-dimensionnelle. L'égalité dans (1.9) peut se déduire de la formule de HOEFFDING (1940) (généralisée dans le théorème 2.1 de QUESADA-MOLINA, 1992) donnée par :

$$\operatorname{cov}(X_1, X_2) = \int_{\mathbb{R}^2} \{F(x,y) - F_1(x)F_2(y)\} \mathrm{d}x\mathrm{d}y,$$

où F est la f.d.r. du vecteur aléatoire (X_1, X_2). En effet, ce résultat appliqué au vecteur (U_1, U_2) nous donne

$$\operatorname{cov}(U_1, U_2) = \int_{[0,1]^2} \{C(u,v) - uv\} \mathrm{d}u\mathrm{d}v,$$

ce qui est le résultat voulu. □

La proposition 1.2.2 n'est en fait valide que pour des copules de dimension 2. Au vu de (1.9), le rho de Spearman s'écrit comme une transformation affine de la «moyenne» de la copule C. C'est pour cette raison que le rho de Spearman est souvent qualifié de «moment» de la copule. On peut donc aisément voir que deux vecteurs aléatoires ayant même rho de Spearman n'ont pas nécessairement la même copule. Au vu des expressions (1.8) et (1.9), le rho de Spearman du vecteur (X_1, X_2) sera également noté $\rho(C)$, où C est la copule du vecteur (X_1, X_2).

Plusieurs extensions multivariées du rho de Spearman ont été proposées dans la littérature. Considérons un vecteur aléatoire d-dimensionnel $\boldsymbol{X}$ dont les f.d.r. marginales $F_1, \ldots, F_d$ sont supposées continues et de copule C. Dans ce contexte, trois généralisations multivariées du rho de Spearman sont suggérées dans Schmid et Schmidt (2007a). Pour comprendre la construction de ces généralisations, on peut voir qu'il est également possible de réécrire le rho de Spearman 2-dimensionnel comme distance normalisée entre la copule C et la copule d'indépendance Π :

$$\rho(C) = \frac{\int_{[0,1]^2} C(u,v)\,\mathrm{d}u\mathrm{d}v - \int_{[0,1]^2} \Pi(u,v)\,\mathrm{d}u\mathrm{d}v}{\int_{[0,1]^2} M(u,v)\,\mathrm{d}u\mathrm{d}v - \int_{[0,1]^2} \Pi(u,v)\,\mathrm{d}u\mathrm{d}v}, \tag{1.10}$$

où les copules M et Π sont définies dans (1.5) et (1.6), respectivement.

Une première généralisation «naturelle» consiste alors à remplacer les copules bivariées C, Π et M dans l'expression (1.10) par leurs versions d-dimensionnelles. Cela conduit au ρ_1 de Schmid et Schmidt (2007a), à savoir,

$$\begin{aligned}\rho_1(C) &= \frac{\int_{[0,1]^d} C(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} - \int_{[0,1]^d} \Pi(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}}{\int_{[0,1]^d} M(\boldsymbol{u})\,\mathrm{d}u - \int_{[0,1]^d} \Pi(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}} \\ &= h(d)\left\{2^d \int_{[0,1]^d} C(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} - 1\right\}, \qquad (1.11)\end{aligned}$$

où $h(d) = (d+1)/(2^d - d - 1)$. Le ρ_2 de Schmid et Schmidt (2007a) est obtenu par extension de l'expression (1.8) au cas de la dimension d en prenant :

$$\begin{aligned}\rho_2(C) &= \frac{\int_{[0,1]^d} \prod_{j=1}^d u_j\,\mathrm{d}C(\boldsymbol{u}) - \int_{[0,1]^d} \Pi(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}}{\int_{[0,1]^d} M(\boldsymbol{u})\,\mathrm{d}u - \int_{[0,1]^d} \Pi(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}} \\ &= h(d)\left\{2^d \mathbb{E}_C(\boldsymbol{U}) - 1\right\} \qquad (1.12) \\ &= h(d)\left\{2^d \int_{[0,1]^d} \bar{C}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} - 1\right\}, \qquad (1.13)\end{aligned}$$

où $\boldsymbol{U}$ est un vecteur aléatoire de fonction de répartition la copule C et $\bar{C}$ est la fonction de survie du vecteur $\boldsymbol{U}$ définie par :

$$\bar{C}(\boldsymbol{u}) = \mathrm{P}(\boldsymbol{U} > \boldsymbol{u}) = \sum_{A \subseteq \{1,\ldots,d\}} (-1)^{|A|} C(\boldsymbol{u}^A), \qquad \boldsymbol{u} \in [0,1]^d. \tag{1.14}$$

Le passage de l'écriture (1.12) à (1.13) se justifie de la façon suivante :

$$\begin{aligned}\int_{[0,1]^d} \prod_{j=1}^{d} u_j \, \mathrm{d}C(\boldsymbol{u}) &= \int_{[0,1]^d} \left\{ \int_{[0,1]^d} \prod_{j=1}^{d} \mathbf{1}(0 \leq x_j \leq u_j) \mathrm{d}\boldsymbol{x} \right\} \mathrm{d}C(\boldsymbol{u}) \\ &= \int_{[0,1]^d} \left\{ \int_{[x_1,1]\times\ldots\times[x_d,1]} \mathrm{d}C(\boldsymbol{u}) \right\} \mathrm{d}\boldsymbol{x} = \int_{[0,1]^d} \bar{C}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.\end{aligned}$$

Notons que lorsque C est la copule de $\boldsymbol{X}$, alors $\bar{C}$ s'exprime en fonction de la copule $\hat{C}$ du vecteur $-\boldsymbol{X}$ (appelée copule de survie) pour tout $\boldsymbol{u} \in [0,1]^d$ par $\bar{C}(\boldsymbol{u}) = \hat{C}(1 - \boldsymbol{u})$. En particulier, on peut voir que $\rho_2(C) = \rho_1(\hat{C})$.

Enfin, une troisième généralisation consiste à considérer la moyenne des rhos de Spearman des marges bivariées de $\boldsymbol{X}$:

$$\begin{aligned}\rho_3(C) &= \binom{d}{2}^{-1} \sum_{1 \leq p < q \leq d} \rho_1(C^{(p,q)}) \\ &= h(2) \binom{d}{2}^{-1} \left\{ 2^2 \sum_{1 \leq p < q \leq d} \int_{[0,1]^d} C(\boldsymbol{u}^{\{p,q\}}) \mathrm{d}\boldsymbol{u} - 1 \right\}, \qquad (1.15)\end{aligned}$$

où pour $1 \leq p \leq q \leq d$, $C^{(p,q)}$ désigne la copule du vecteur (X_p, X_q). Pour $d = 2$, on pourra remarquer que $\rho_1(C) = \rho_2(C) = \rho_3(C)$.

1.2.2 Tau de Kendall et autres mesures de concordances

Lorsque l'on dispose d'un vecteur aléatoire bivarié, une autre façon fréquemment utilisée pour mesurer la dépendance entre ses composantes consiste à quantifier la «probabilité de concordance» associée au vecteur aléatoire.

Définition 1.2.1. *Deux vecteurs $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ sont dit concordants si $(x_1 - y_1)(x_2 - y_2) > 0$ et discordants si $(x_1 - y_1)(x_2 - y_2) < 0$.*

Considérons une copie indépendante $\boldsymbol{Y} = (Y_1, Y_2)$ de $\boldsymbol{X}$, c'est-à-dire un vecteur aléatoire de même loi que $\boldsymbol{X}$ mais indépendant de $\boldsymbol{X}$. Le tau de Kendall est défini comme étant la différence entre la probabilité que $\boldsymbol{X}$ et $\boldsymbol{Y}$ soient concordants et la probabilité que $\boldsymbol{X}$ et $\boldsymbol{Y}$ soient discordants. Plus formellement,

$$\tau(X_1, X_2) = \mathrm{P}\{(X_1 - Y_1)(X_2 - Y_2) > 0\} - \mathrm{P}\{(X_1 - Y_1)(X_2 - Y_2) < 0\}. \qquad (1.16)$$

Dans la proposition 5.29 de McNeil et coll. (2005), on trouve que le tau de Kendall peut également s'exprimer en terme de la copule C du vecteur $\boldsymbol{X}$, toujours dans le cas où les f.d.r. marginales de $\boldsymbol{X}$ sont continues. En effet,

$$\tau(X_1, X_2) = 4 \int_{[0,1]^2} C(\boldsymbol{u}) \mathrm{d}C(\boldsymbol{u}) - 1. \tag{1.17}$$

À titre indicatif, des extensions multivariées du tau de Kendall (voir p. ex. Joe, 1997; Quessy et coll., 2013) ont été proposées, mais nous ne les étudierons pas dans ce travail.

Notons enfin que le rho de Spearman défini dans (1.7) peut s'écrire également comme mesure de concordance. On peut trouver dans le théorème 5.1.6 de Nelsen (2006) que l'expression (1.2.2) peut s'écrire :

$$\rho(X_1, X_2) = 3 \left[\mathrm{P}\{(X_1 - Y_1)(X_2 - Z_2) > 0\} - \mathrm{P}\{(X_1 - Y_1)(X_2 - Z_2) < 0\} \right],$$

où (Y_1, Y_2) et (Z_1, Z_2) sont des copies indépendantes de (X_1, X_2). Bien que différents dans le cas général, le rho de Spearman et le tau de Kendall partagent des propriétés communes. Ce sont toutes deux des mesures symétriques, prenant leurs valeurs sur $[-1, 1]$, valant 1 lorsque X_1 et X_2 sont dites comonotoniques (dépendance positive parfaite), c'est-à-dire, lorsque le vecteur (X_1, X_2) admet M pour copule et -1 lorsque qu'elles sont dites antimonotoniques (dépendance négative parfaite), c'est-à-dire, lorsque le vecteur (X_1, X_2) admet W pour copule.

Dans le cas de nombreuses familles de copules paramétrées par un réel θ, il existe une bijection entre θ et le rho de Spearman ou le tau de Kendall. Dans certain cas, la relation est explicite et il est alors possible de calculer le rho de Spearman ainsi que le tau de Kendall d'un vecteur aléatoire à partir du paramètre de la copule associée au vecteur aléatoire et vice versa. Par exemple, dans le cas des copules de Gumbel–Hougaard, on a $\tau = 1 - 1/\theta$. Pour les copules de Clayton, $\tau = \theta/(\theta + 2)$. Dans d'autre cas, la bijection entre θ et τ et/ou ρ ne s'écrit pas de façon explicite, et on pourra avoir recours à une approximation numérique.

D'autres mesures de la dépendances sont également présentes dans la littérature. On pourra par exemple retrouver le bêta de Blomqvist, étudié et généralisé dans Schmid et Schmidt (2007b), défini par

$$\beta(X_1, X_2) = 4C(1/2, 1/2) - 1,$$

où encore le gamma de Gini

$$\gamma(X_1, X_2) = 2 \int_{[0,1]^2} \{|u + v - 1| - |u - v|\} \, \mathrm{d}C(u, v),$$

où C est la copule du vecteur (X_1, X_2).

CHAPITRE 2

Copule empirique et processus séquentiels

L'objectif de ce chapitre est de définir les outils nécessaires à la construction des tests abordés dans la partie 2. Nous rappellerons les résultats prédominants que l'on trouve dans la littérature en matière de *processus de copule empirique séquentiel* pour des données indépendantes ou α-*mélangeantes*.

Deux estimateurs classiques de la copule seront présentés. Le comportement asymptotique du processus de copule empirique et ainsi que du processus de copule empirique séquentiel seront détaillés. Des méthodes de rééchantillonnages associées seront également abordées, pour le cas indépendant et pour le cas fortement mélangeant.

Sommaire

2.1 Copule empirique

Dans un souci de clarté, le symbole $\rightsquigarrow$ est utilisé dans la suite de ce travail, pour parler de la convergence faible au sens de la définition 1.3.3 de VAN DER VAART et WELLNER (2000, page 18). Pour $T \subseteq \mathbb{R}^d$, l'ensemble des fonctions réelles bornées sur T, muni de la métrique uniforme, est noté $\ell^\infty(T)$.

Soient $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ des vecteurs aléatoires issus d'un processus strictement stationnaire $(\boldsymbol{X}_t)_{t \in \mathbb{Z}}$ d-dimensionnel, tel que, pour tout $t \in \mathbb{Z}$, $\boldsymbol{X}_t$ admette F pour f.d.r., $F_1, \ldots, F_d$ pour f.d.r. marginales, supposées continues, et C pour copule. Nous nous plaçons dans ce travail dans un contexte d'estimation non paramétrique. Les f.d.r. F,$F_1, \ldots, F_d$ et la copule C sont inconnues.

Considérons dans la suite les vecteurs aléatoires non-observables $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$ obtenus à partir des $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ par la transformation $U_{ij} = F_j(X_{ij})$, $i \in \{1, \ldots, n\}$ et $j \in \{1, \ldots, d\}$. Comme vu au chapitre précédent, nous savons alors que les vecteurs aléatoires $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$ admettent C pour f.d.r.

Si les vecteurs $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$ étaient observables, on pourrait estimer la f.d.r. inconnue C par la f.d.r. empirique calculée à partir de l'échantillon $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$. Les f.d.r. marginales $F_1, \ldots, F_d$ étant inconnues, nous allons les estimer en remplaçant les f.d.r. inconnues par leurs versions empiriques.

Pour tout $i \in \{1, \ldots, n\}$, notons

$$\hat{\boldsymbol{U}}_i^{1:n} = (\hat{U}_{i1}^{1:n}, \ldots, \hat{U}_{id}^{1:n}),$$

où, pour $j \in \{1, \ldots, d\}$,

$$\hat{U}_{ij}^{1:n} = \frac{n}{n+1} F_{1:n,j}(X_{ij}), \tag{2.1}$$

et où $F_{1:n,j}$ désigne la fonction de répartition empirique obtenue à partir de l'échantillon $X_{1j}, \ldots, X_{nj}$. Cette dernière est donnée par

$$F_{1:n,j}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(X_{ij} \leq x), \qquad x \in \mathbb{R}.$$

Dans cette section, les estimateurs sont indicés de façon «non-naturelle» par $1{:}n$ plutôt que par n. Cette notation sera justifiée dans la section 2.3.

Les vecteurs $\hat{\boldsymbol{U}}_1^{1:n}, \ldots, \hat{\boldsymbol{U}}_n^{1:n}$ sont parfois qualifiés de pseudo-observations de la copule C. Le facteur $n/(n+1)$ apparaissant dans la définition des pseudo-observations (2.1) est généralement introduit pour éviter des problèmes aux frontières dans la théorie des statistiques de rangs, notamment lors de l'utilisation de fonctions scores non nécessairement définies sur les bords de $[0,1]^d$. On peut par exemple retrouver l'utilisation de telles fonctions scores dans Hájek et coll. (1967) ou encore Lombard (1987).

En partant de l'écriture (2.1), on peut voir que, lorsqu'il n'y a pas d'ex-æquo dans les échantillons $X_{1j}, \ldots, X_{nj}$ pour $j \in \{1, \ldots, d\}$ les vecteurs des pseudo-observations peuvent s'écrire en terme des rangs des échantillons $X_{1j}, \ldots, X_{nj}$ de la façon suivante (ce qui justifie leur autre appellation courante de vecteurs de rangs renormalisés) :

$$\hat{\boldsymbol{U}}_i^{1:n} = \frac{1}{n+1} (R_{i1}^{1:n}, \ldots, R_{id}^{1:n}),$$

où, pour $i \in \{1, \ldots, n\}$ et $j \in \{1, \ldots, d\}$, $R_{ij}^{1:n}$ désigne le rang de X_{ij} dans l'échantillon $X_{1j}, \ldots, X_{nj}$.

Il est à noter que lorsque les vecteurs $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ sont indépendants et identiquement distribués (i.i.d.) et que leurs marges sont continues, les rangs $R_{ij}^{1:n}$ pour $i \in \{1, \ldots, n\}$ et $j \in \{1, \ldots, d\}$ sont parfaitement définis. Dans le cas de dépendance sérielle dans les données, la continuité des marges ne suffit plus à garantir l'absence d'égalité dans les échantillons $X_{1j}, \ldots, X_{nj}$. La notion de rang doit alors être redéfinie (p. ex. rangs minimaux, maximaux, moyens). C'est par exemple le cas d'observations issues des processus *maxima par bloc* étudiés dans BÜCHER et SEGERS (2013). Dans BÜCHER et KOJADINOVIC (2013), la condition suivante est faite :

Condition 2.1.1. *Avec probabilité un, il n'y a pas d'ex-æquo parmi les j^e composantes $X_{1j}, \ldots, X_{nj}$ des vecteurs aléatoires $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$, quel que soit $j \in \{1, \ldots, d\}$.*

Un estimateur non paramétrique de la copule C est alors la f.d.r. empirique des pseudo-observations. Introduit dans RÜSCHENDORF (1976) et DEHEUVELS (1979, 1981), il est défini, pour $\boldsymbol{u} \in [0,1]^d$, par :

$$C_{1:n}(\boldsymbol{u}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n} \leq \boldsymbol{u}). \tag{2.2}$$

La copule empirique au sens de la définition (2.2), diffère légèrement de l'estimateur «naturel» de la copule que l'on trouve par exemple dans les travaux de FERMANIAN et coll. (2004) ou encore dans SEGERS (2012). Cet estimateur est basé sur l'écriture (1.4) de la copule C. La stratégie est d'estimer F ainsi que les fonctions quantiles $F_1^{-1}, \ldots, F_d^{-1}$ par leurs versions empiriques classiques $F_{1:n}$ et $F_{1:n,1}^{-1}, \ldots, F_{1:n,d}^{-1}$, données par :

$$F_{1:n}(\boldsymbol{x}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(\boldsymbol{X}_i \leq \boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

et, pour $j \in \{1, \ldots, d\}$, $F_{1:n,j}^{-1}$ est l'inverse empirique généralisée de $F_{1:n,j}$:

$$F_{1:n,j}^{-1}(u) = \inf\{v \in [0,1] : F_{1:n,j}(v) \geq u\} \qquad u \in [0,1].$$

L'estimateur en question de la copule C est alors donné par

$$\hat{C}_{1:n}(\boldsymbol{u}) = F_{1:n}(F_{1:n,1}^{-1}(u_1), \ldots, F_{1:n,d}^{-1}(u_d)), \qquad \boldsymbol{u} \in [0,1]^d. \tag{2.3}$$

La différence entre les deux estimateurs $C_{1:n}$ et $\hat{C}_{1:n}$ peut être majorée par d/n dans le cas de données sériellement indépendantes ou satisfaisant la condition 2.1.1.

Lorsque la condition 2.1.1 n'est pas satisfaite, dans certaines situations, il est néanmoins possible de quantifier la différence entre ces deux estimateurs. Pour cela, considérons le processus empirique des vecteurs aléatoires non observables $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$, à savoir,

$$\mathbb{Z}_n^0(\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{\mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}) - C(\boldsymbol{u})\}, \qquad \boldsymbol{u} \in [0,1]^d. \tag{2.4}$$

Le résultat suivant peut être trouvé dans le lemme A.2 de Bücher et Segers (2013) :

Lemme 2.1.1. *Dès que les vecteurs aléatoires $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$ sont tels que le processus $\mathbb{Z}_n^0$ converge faiblement dans $\ell^\infty([0,1]^d)$ vers un processus dont les trajectoires sont continues presque sûrement, alors*

$$\sup_{\boldsymbol{u} \in [0,1]^d} |\hat{C}_{1:n}(\boldsymbol{u}) - C_{1:n}(\boldsymbol{u})| = o_P(n^{-1/2}).$$

Ainsi, que ce soit dans le cas de données indépendantes, satisfaisant la condition 2.1.1 ou encore satisfaisant la condition du lemme 2.1.1, $C_{1:n}$ et $\hat{C}_{1:n}$ ont le même comportement asymptotique.

Bien que peut-être plus naturel, l'estimateur $\hat{C}_{1:n}$ a une forme plus complexe que l'estimateur $C_{1:n}$, dont l'évaluation ne nécessite que le calcul des rangs de l'échantillon. L'ensemble de nos résultats de la partie 2 seront fondés, pour cette raison, uniquement sur $C_{1:n}$.

2.2 Processus de copule empirique

2.2.1 Comportement asymptotique du processus

De façon assez naturelle, nous nous intéressons au comportement asymptotique du processus de copule empirique que nous présentons dans cette section dans sa forme standard :

$$\mathbb{C}_n^0(\boldsymbol{u}) = \sqrt{n}\{C_{1:n}(\boldsymbol{u}) - C(\boldsymbol{u})\}, \qquad \boldsymbol{u} \in [0,1]^d. \tag{2.5}$$

En dimension $d = 2$, le processus $\mathbb{C}_n^0$ a été étudié, entre autres, par Rüschendorf (1976), Gänssler et Stute (1987), van der Vaart et Wellner (2000), Fermanian et coll. (2004), van der Vaart et Wellner (2007), Segers (2012) et Bücher et Volgushev (2013). Dans le cas où les vecteurs $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ sont i.i.d., le comportement asymptotique du processus $\mathbb{C}_n^0$ est établi par van der Vaart et Wellner (2000, p. 389) dans l'espace $\ell^\infty([p,q]^2)$

avec $0 < p < q < 1$ et sous la condition d'existence et de continuité des dérivées partielles de la copule sur $[p, q]^2$. Le résultat provient d'une application de la méthode Delta fonctionnelle. Le théorème 3 de l'article de FERMANIAN et coll. (2004) élargit ce résultat à l'ensemble $\ell^\infty([0,1]^2)$ sous l'hypothèse que les dérivées partielles de la copule existent et sont continues sur $[0,1]^2$. Néanmoins, pour un certain nombre de copules utilisées en pratique, les dérivées partielles n'existent pas sur l'ensemble $[0,1]^2$. Dans sa proposition 3.1, SEGERS (2012) montre la convergence du processus de copule empirique d-dimensionnel $\mathbb{C}_n^0$ dans $\ell^\infty([0,1]^d)$ sous la condition suivante, «peu» restrictive (comme nous l'expliquerons après) :

Condition 2.2.1. *Pour tout $j \in \{1, \dots, d\}$, les dérivées partielles $\dot{C}_j = \partial C / \partial u_j$ existent et sont continues sur l'ensemble $V_j = \{\boldsymbol{u} \in [0,1]^d, u_j \in (0,1)\}$.*

Un grand nombre de copules étudiées dans la littérature satisfait la condition 2.2.1. SEGERS (2012) montre que c'est le cas par exemple des copules Archimédiennes, ou encore des copules de valeurs extrêmes (les définitions de ces classes de copules peuvent respectivement être trouvées dans le chapitre 4 et la section 3.3.4 de NELSEN, 2006). En particulier c'est le cas des copules d-dimmensionnelles de Gumbel–Hougaard, Clayton, Frank, normales ou de Student. Néanmoins, ce n'est pas le cas, en dimension 2, des copules de comonotonicité et d'antimonotonicité M et W formant les bornes de Fréchet-Hoeffding définies dans (1.5). Par conséquent, ce n'est pas le cas non plus des copules 2-dimensionnelles de la forme $\delta C_1 + (1-\delta) C_2$, avec $\delta \in [0,1]$ quand C_1 ou bien C_2 est l'une des copules M ou W. Ces exemples sont détaillés dans la section 5 de SEGERS (2012). Bien qu'excluant un certain nombre de copules, la condition 2.2.1 n'est pas restrictive dans le sens où elle est nécessaire pour garantir que la loi limite du processus $\mathbb{C}_n^0$ existe ponctuellement et que ses trajectoires sont continues presque sûrement.

Dans la suite, pour tout $j \in \{1, \dots, d\}$, nous adopterons la convention que $\dot{C}_j = 0$ sur l'ensemble $\{\boldsymbol{u} \in [0,1]^d ; u_j = 0 \text{ ou } u_j = 1\}$. La proposition 3.1 apparaissant dans SEGERS (2012) est la suivante :

Proposition 2.2.1. *Soient $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ des vecteurs aléatoires i.i.d., de dimension d, de copule C et dont les marges sont continues. Alors, sous la condition 2.2.1 on a*

$$\sup_{\boldsymbol{u} \in [0,1]^d} |\mathbb{C}_n^0(\boldsymbol{u}) - \widetilde{\mathbb{C}}_n^0(\boldsymbol{u})| \xrightarrow{\mathrm{P}} 0,$$

où, pour $\boldsymbol{u} \in [0,1]^d$,

$$\widetilde{\mathbb{C}}_n^0(\boldsymbol{u}) = \mathbb{Z}_n^0(\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u}) \mathbb{Z}_n^0(\boldsymbol{u}^{\{j\}}), \tag{2.6}$$

avec $\mathbb{Z}_n^0$ le processus empirique défini dans (2.4) et le vecteur $\boldsymbol{u}^{\{j\}}$ est défini dans (1.1) : il désigne le vecteur composé que de 1 sauf à la composante j pour laquelle $u_j^{\{j\}} = u_j$.

Dans le cadre de vecteurs aléatoires i.i.d, un résultat bien connu est que le processus $\mathbb{Z}_n^0$ converge en loi dans $\ell^\infty([0,1]^d)$ vers un pont Brownien $\mathbb{Z}_C^0$ dont les trajectoires sont continues presque sûrement, c'est-à-dire, un processus Gaussien tendu, centré, et de covariance donnée par

$$\mathrm{cov}\{\mathbb{Z}_C^0(\boldsymbol{u}), \mathbb{Z}_C^0(\boldsymbol{v})\} = C(\boldsymbol{u} \wedge \boldsymbol{v}) - C(\boldsymbol{u})C(\boldsymbol{v}), \qquad \text{pour tout } \boldsymbol{u}, \boldsymbol{v} \in [0,1]^d, \tag{2.7}$$

où, pour deux vecteurs $\boldsymbol{u}$ et $\boldsymbol{v}$ de $[0,1]^d$, $\boldsymbol{u} \wedge \boldsymbol{v}$ désigne le minimum composante à composante, c'est-à-dire, $\boldsymbol{u} \wedge \boldsymbol{v} = (\min(u_1, v_1), \ldots, \min(u_d, v_d))$.

Une conséquence immédiate de la proposition 2.2.1 et du théorème des applications continues est que, sous les mêmes conditions que la proposition 2.2.1,

$$\mathbb{C}_n^0(\boldsymbol{u}) \rightsquigarrow \mathbb{C}_C^0(\boldsymbol{u}) \text{ dans } \ell^\infty([0,1]^d),$$

où le processus $\mathbb{C}_C^0$ est défini par

$$\mathbb{C}_C^0(\boldsymbol{u}) = \mathbb{Z}_C^0(\boldsymbol{u}) - \sum_{j=1}^d \dot{C}_j(\boldsymbol{u}) \mathbb{Z}_C^0(\boldsymbol{u}^{\{j\}}), \qquad \boldsymbol{u} \in [0,1]^d. \tag{2.8}$$

Dans la pratique, il n'est souvent pas judicieux de supposer l'indépendance des vecteurs aléatoires $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$. Dans le cas des séries temporelles, cette hypothèse est même très généralement inadéquate. Les articles de Bücher et Volgushev (2013), Bücher et Kojadinovic (2013) et Bücher (2013) s'intéressent en particulier au cas de données *α-mélangeantes* au sens de Rosenblatt (1956).

Définition 2.2.1. *Considérons la suite de vecteurs aléatoires d-dimensionnels $(\mathbf{Y}_i)_{i \in \mathbb{Z}}$. Pour $s, t \in \mathbb{Z} \cup \{-\infty, +\infty\}$, soit $\mathcal{F}_s^t$ la σ-algèbre générée par les vecteurs $\mathbf{Y}_i$, $s \leq i \leq t$, c'est-à-dire, $\mathcal{F}_s^t = \sigma(\mathbf{Y}_i, s \leq i \leq t)$. Le coefficient de mélange fort de* Rosenblatt *(1956) est défini par*

$$\alpha_r = \sup_{t \in \mathbb{Z}} \sup_{A \in \mathcal{F}_{-\infty}^t, B \in \mathcal{F}_{t+r}^{+\infty}} |P(A \cap B) - P(A)P(B)|.$$

Le processus $(\mathbf{Y}_i)_{i \in \mathbb{Z}}$ est dit α-mélangeant (ou fortement mélangeant, au sens du coefficient α) si $\alpha_r \to 0$ quand $r \to +\infty$.

Dans la littérature, d'autres coefficients de mélange sont présents, donnant lieu à d'autres processus fortement mélangeants. On mentionnera à titre d'exemple les processus β-mélangeants (processus absolument réguliers) ou bien encore ϕ-mélangeants (processus uniformément mélangeants). On pourra trouver leur définition dans DEHLING et PHILIPP (2002, définition 3.1). Néanmoins, les cas du β-mélange et du ϕ-mélange impliquent tous les deux l'α-mélange. Par conséquent, il s'agit de classes de processus plus restrictives et donc moins intéressantes dans le sens où ils donnent lieu à des résultats moins généraux. Comme corollaire des travaux de BÜCHER et VOLGUSHEV (2013, corollaire 2.5) et de BÜCHER (2013, théorème 1), on a le résultat suivant :

Proposition 2.2.2. *Soient $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ des vecteurs aléatoires de dimension d, issus d'un processus strictement stationnaire $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ d-dimensionnel, fortement mélangeant de coefficient de mélange $\alpha_r = O(r^{-a})$, $a > 1$, dont les marges sont continues et de copule C. Alors le résultat de la proposition 2.2.1 reste valide, la covariance du processus $\mathbb{Z}_C$ étant donnée dans ce cas-ci par*

$$\operatorname{cov}\{\mathbb{Z}_C^0(\boldsymbol{u}), \mathbb{Z}_C^0(\boldsymbol{v})\} = \sum_{k\in\mathbb{Z}} \operatorname{cov}\{\mathbf{1}(\boldsymbol{U}_0 \leq \boldsymbol{u}), \mathbf{1}(\boldsymbol{U}_k \leq \boldsymbol{v})\}, \qquad \boldsymbol{u}, \boldsymbol{v} \in [0,1]^d.$$

2.2.2 Rééchantillonnage du processus de copule empirique dans le cas de données indépendantes

Le processus $\mathbb{C}_n^0$ n'étant pas calculable puisque la copule C est inconnue, il est important en pratique d'être capable de le rééchantillonner, c'est-à-dire, de construire des processus $\mathbb{C}_n^{0,(1)}, \mathbb{C}_n^{0,(2)}, \dots$ qui vont converger faiblement, dans un sens approprié, vers des copies indépendantes du processus $\mathbb{C}_C^0$. De tels rééchantillonnages serviront notamment dans la construction de tests pour la détection de ruptures dans la copule des observations.

Plaçons-nous dans cette section dans le cas de données i.i.d. FERMANIAN et coll. (2004) proposent dans leur théorème 5, de rééchantillonner le processus $\hat{\mathbb{C}}_n^0(\boldsymbol{u}) = \sqrt{n}\{\hat{C}_{1:n}(\boldsymbol{u}) - C(\boldsymbol{u})\}$, $\boldsymbol{u} \in [0,1]^d$, où $\hat{C}_{1:n}$ est la copule empirique définie dans (2.3), à l'aide d'un bootstrap multinomial classique. Pour ce faire on tire de façon équiprobable, avec remise, n vecteurs aléatoires dans l'échantillon $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ et on calcule la copule empirique $\hat{C}_{1:n}^b$ de l'échantillon ainsi formé. Alors FERMANIAN et coll. (2004) montrent que, conditionnellement aux données, le processus $\{\sqrt{n}\{\hat{C}_{1:n}^b(\boldsymbol{u}) - \hat{C}_{1:n}(\boldsymbol{u})\}, \boldsymbol{u} \in [0,1]^2\}$ *converge faiblement en probabilité* (au sens de la section 2.2.3 de KOSOROK, 2008) dans $\ell^\infty([0,1]^2)$ vers le processus $\mathbb{C}_C$ défini dans (2.8).

Une autre façon de rééchantillonner le processus $\mathbb{C}_n^0$ va consister à rééchantillonner le processus $\mathbb{Z}_n^0$. En effet, la loi limite du processus $\widetilde{\mathbb{C}}_n^0$ défini dans l'équation (2.6) s'écrit comme fonctionnelle de la limite faible du processus $\mathbb{Z}_n^0$. Dans le cas où les vecteurs $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ sont i.i.d., Scaillet (2005) et Rémillard et Scaillet (2009) proposent de rééchantillonner $\mathbb{Z}_n^0$ en utilisant une approche à base de *multiplicateurs* dans l'esprit du chapitre 2.9 de van der Vaart et Wellner (2000). Bücher et Dette (2010) ont comparé les performances des différentes techniques de rééchantillonnage utilisées dans la littérature et ont conclu que le rééchantillonnage offrant les meilleurs propriétés à tailles finies était celui proposé par Rémillard et Scaillet (2009). Cette technique de rééchantillonnage a été reprise par Segers (2012), qui a montré sa validité asymptotique sous la condition 2.2.1.

Définition 2.2.2. *Une suite de variables aléatoires* $(\xi_i)_{i\in\mathbb{Z}}$ *est appelée suite i.i.d. de multiplicateurs, si elle est i.i.d., indépendante des observations* $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$*, et satisfait la condition* (**M0**) *suivante :*

(M0) $\mathbb{E}_\xi(\xi_0) = 0$, $\mathrm{var}_\xi(\xi_0) = 1$ *et* $\int_0^\infty \{\mathrm{P}(|\xi_0| > x)\}^{1/2} \mathrm{d}x < \infty$.

Des exemples types de telles suites i.i.d. de multiplicateurs pourront être des suites i.i.d. de variables aléatoires normales centrées réduites ou bien encore de Rademacher (c'est-à-dire que $\mathrm{P}(\xi_0 = 1) = \mathrm{P}(\xi_0 = -1) = 1/2$).

Pour un entier M, considérons M copies indépendantes $(\xi_i^{(1)})_{i\in\mathbb{Z}}, \dots, (\xi_i^{(M)})_{i\in\mathbb{Z}}$ d'une même suite i.i.d. de multiplicateurs, et définissons, pour $m \in \{1, \dots, M\}$, les processus évaluables

$$\begin{aligned}\mathbb{Z}_n^{0,(m)}(\boldsymbol{u}) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i^{(m)} \{\mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n} \le \boldsymbol{u}) - C_{1:n}(\boldsymbol{u})\} \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\xi_i^{(m)} - \bar{\xi}_n^{(m)}\right) \mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n} \le \boldsymbol{u}), \qquad \boldsymbol{u} \in [0,1]^d,\end{aligned}$$

où $\bar{\xi}_n^{(m)} = n^{-1} \sum_{i=1}^n \xi_i^{(m)}$. Dans Rémillard et Scaillet (2009), il est montré que

$$\left(\mathbb{Z}_n^0, \mathbb{Z}_n^{0,(1)}, \dots, \mathbb{Z}_n^{0,(M)}\right) \rightsquigarrow \left(\mathbb{Z}_C^0, \mathbb{Z}_C^{0,(1)}, \dots, \mathbb{Z}_C^{0,(M)}\right) \quad \text{dans } \{\ell^\infty([0,1]^d)\}^{M+1},$$

où $\mathbb{Z}_C^{0,(1)}, \dots, \mathbb{Z}_C^{0,(M)}$ sont des copies indépendantes du processus $\mathbb{Z}_C^0$. Pour n grand, les processus $\mathbb{Z}_n^{0,(1)}, \dots, \mathbb{Z}_n^{0,(M)}$ peuvent être vus comme des copies «presque» indépendantes du processus $\mathbb{Z}_n^0$. Ils ne sont pas indépendants de $\mathbb{Z}_n^0$ au sens strict, mais ils sont néanmoins non-corrélés et leurs lois limites sont quant à elles indépendantes du processus $\mathbb{Z}_C^0$ puisqu'il s'agit alors de processus gaussiens dont la structure de covariance est donnée dans (2.7).

Dans la suite de ce travail, pour montrer la validité asymptotique des méthodes de rééchantillonnage, on se focalisera uniquement sur une telle approche fondée sur une convergence faible «non conditionnelle». Une discussion plus approfondie sur les différences entre l'approche «conditionnelle» classique et l'approche «non conditionnelle» mentionnée ci-dessus, peut être trouvée dans la remarque 3.2 de BÜCHER et KOJADINOVIC (2013).

Afin de mettre en place un rééchantillonnage à base de multiplicateurs pour le processus $\mathbb{C}_n^0$, il est nécessaire d'estimer les dérivées partielles inconnues $\dot{C}_j$. Nous considérons les estimateurs $\dot{C}_j^{1:n}$ de $\dot{C}_j$ satisfaisant la condition suivante, proposée par SEGERS (2012) :

Condition 2.2.2. *Il existe une constante $K > 0$ telle que $|\dot{C}_j^{1:n}(\boldsymbol{u})| < K$ pour tout $j \in \{1, \dots, d\}$, $n \geq 1$ et tout $\boldsymbol{u} \in [0,1]^d$. De plus, pour tout $\delta \in]0, 1/2[$ et $j \in \{1, \dots, d\}$,*

$$\sup_{\substack{\boldsymbol{u} \in [0,1]^d \\ u_j \in [\delta, 1-\delta]}} |\dot{C}_j^{1:n}(\boldsymbol{u}) - \dot{C}_j(\boldsymbol{u})| \xrightarrow{\mathrm{P}} 0.$$

Considérons alors le processus

$$\mathbb{C}_n^{0,(m)}(\boldsymbol{u}) = \mathbb{Z}_n^{0,(m)}(\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_j^{1:n}(\boldsymbol{u}) \mathbb{Z}_n^{0,(m)}(\boldsymbol{u}^{\{j\}}), \qquad \boldsymbol{u} \in [0,1]^d.$$

Dans sa proposition 3.2, SEGERS (2012) montre le résultat suivant :

Proposition 2.2.3. *Sous les conditions 2.2.1 et 2.2.2, on a :*

$$(\mathbb{C}_n^0, \mathbb{C}_n^{0,(1)}, \dots, \mathbb{C}_n^{0,(M)}) \rightsquigarrow (\mathbb{C}_C^0, \mathbb{C}_C^{0,(1)}, \dots, \mathbb{C}_C^{0,(M)}), \qquad \textit{dans } \{\ell^\infty([0,1]^d)\}^{M+1},$$

où $\mathbb{C}_C^{0,(1)}, \dots, \mathbb{C}_C^{0,(M)}$ sont des copies indépendantes du processus $\mathbb{C}_C^0$ défini dans (2.8).

Plusieurs estimateurs des dérivées partielles vérifiant la condition 2.2.2 ont été étudiés. RÉMILLARD et SCAILLET (2009) proposent d'estimer $\dot{C}_j$ pour $j \in \{1, \dots, d\}$ par simples différences finies :

$$\dot{C}_j^{1:n}(\boldsymbol{u}) = \frac{C_{1:n}(\boldsymbol{u} + h_n \boldsymbol{e}_j) - C_{1:n}(\boldsymbol{u} - h_n \boldsymbol{e}_j)}{2h_n}, \qquad \boldsymbol{u} \in [0,1]^d, \qquad (2.9)$$

où $\boldsymbol{e}_j$ est le j^e vecteur unité de la base canonique de $\mathbb{R}^d$ et $h_n = n^{-1/2}$. Dans KOJADINOVIC et coll. (2011), une version similaire est proposée, où le dénominateur dans (2.9) est remplacé par $\min(u_j + h_n, 1) - \max(u_j - h_n, 0)$. La version proposée par KOJADINOVIC et coll. (2011) a l'avantage de converger uniformément sur $[0,1]^d$ dès que $\dot{C}_j$ est continue sur $[0,1]^d$. On pourra trouver un autre exemple d'estimateur des dérivées partielles satisfaisant la condition 2.2.2 dans l'exemple 2 de BÜCHER et RUPPERT (2013).

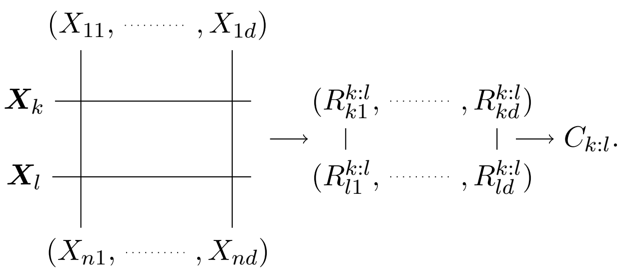

FIGURE 2.1 – Construction de la copule empirique du sous-échantillon $\boldsymbol{X}_k, \dots, \boldsymbol{X}_l$ pour $1 \leq k \leq l \leq n$.

2.3 Processus de copule empirique séquentiel

2.3.1 Comportement asymptotique des processus

Dans cette section, nous allons focaliser notre attention sur des processus doublement séquentiels (de sommes partielles), c'est-à-dire évalués à partir d'une partie de l'échantillon et non de l'échantillon complet. Nous nous plaçons dans le cas où les marges des vecteurs aléatoires $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ sont continues et sous la condition 2.1.1 qui garantit l'unicité des rangs des échantillons $X_{1j}, \dots, X_{nj}$ pour $j \in \{1, \dots, d\}$.

La copule empirique calculée à partir du sous-échantillon $\boldsymbol{X}_k \dots, \boldsymbol{X}_l$, $1 \leq k \leq l \leq n$, est donnée, pour $\boldsymbol{u} \in [0,1]^d$, par :

$$C_{k:l}(\boldsymbol{u}) = \frac{1}{l-k+1} \sum_{i=k}^{l} \mathbf{1}(\hat{\boldsymbol{U}}_i^{k:l} \leq \boldsymbol{u}), \tag{2.10}$$

avec la convention que $C_{k:l} = 0$ dès que $k > l$, et où $\hat{\boldsymbol{U}}_k^{k:l}, \dots, \hat{\boldsymbol{U}}_l^{k:l}$ désignent les pseudo-observations du sous-échantillon $\boldsymbol{X}_k, \dots, \boldsymbol{X}_l$. Elles sont définies de la façon suivante :

$$\hat{\boldsymbol{U}}_i^{k:l} = (\hat{U}_{i1}^{k:l}, \dots, \hat{U}_{id}^{k:l}) = \frac{1}{l-k+2}(R_{i1}^{k:l}, \dots, R_{id}^{k:l}), \qquad i \in \{k, \dots, l\}, \tag{2.11}$$

avec $R_{ij}^{k:l}$ le rang de X_{ij} dans $X_{kj}, \dots, X_{lj}$ pour tout $i \in \{k, \dots, l\}$ et $j \in \{1, \dots, d\}$. La figure 2.3.1 illustre le fait que les définitions ci-dessus ne sont rien d'autre qu'une adaptation immédiate des définitions de la section précédente au sous-échantillon $\boldsymbol{X}_k, \dots, \boldsymbol{X}_l$.

La quantité $C_{k:l}$ est la f.d.r. empirique des pseudo-observations du sous-échantillon $\boldsymbol{X}_k, \dots, \boldsymbol{X}_l$. Les rangs $R_{ij}^{k:l}$ pour $1 \leq k \leq l \leq n$, $i \in \{1, \dots, n\}$ et $j \in \{1, \dots, d\}$ peuvent s'écrire également comme

$$R_{ij}^{k:l} = (l - k + 1) F_{k:l,j}(X_{ij}), \tag{2.12}$$

où $F_{k:l,j}$ désigne la f.d.r. empirique, calculée à partir du sous-échantillon $X_{kj}, \dots, X_{lj}$ pour $1 \leq k \leq l \leq n$ et $j \in \{1, \dots, d\}$, c'est-à-dire,

$$F_{k:l,j}(x) = \frac{1}{l - k + 1} \sum_{i=k}^{l} \mathbf{1}(X_{ij} \leq x), \qquad x \in \mathbb{R}.$$

Nous pouvons à présent définir le processus de copule empirique doublement séquentiel. Initialement proposé dans BÜCHER et KOJADINOVIC (2013), il est défini sur $\Delta \times [0,1]^d$, $\Delta = \{(s,t) \in [0,1]^{d+1} : s \leq t\}$, par

$$\mathbb{C}_n(s,t,\boldsymbol{u}) = \sqrt{n}\, \lambda_n(s,t) \{C_{\lfloor ns \rfloor + 1:\lfloor nt \rfloor}(\boldsymbol{u}) - C(\boldsymbol{u})\}, \tag{2.13}$$

où pour $(s,t) \in \Delta$, $\lambda_n(s,t) = (\lfloor nt \rfloor - \lfloor ns \rfloor)/n$, et pour $x \in \mathbb{R}$, $\lfloor x \rfloor$ désigne la partie entière de x. Ce dernier peut également s'écrire de la façon suivante :

$$\mathbb{C}_n(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \left\{ \mathbf{1}(\hat{\boldsymbol{U}}_i^{\lfloor ns \rfloor + 1:\lfloor nt \rfloor} \leq \boldsymbol{u}) - C(\boldsymbol{u}) \right\}, \tag{2.14}$$

$(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$.

L'intérêt principal du processus doublement séquentiel ci-dessus est qu'il va permettre de simplifier l'étude asymptotique des statistiques de test pour la détection de ruptures dans la copule, définies dans la deuxième partie de ce travail. En effet, ces dernières pourront s'écrire comme différences pondérées entre le processus séquentiel, $(s,\boldsymbol{u}) \mapsto \mathbb{C}_n(0,s,\boldsymbol{u})$ et le processus séquentiel, $(s,\boldsymbol{u}) \mapsto \mathbb{C}_n(s,1,\boldsymbol{u})$.

Notons que le processus séquentiel $\mathbb{C}_n(0,.,.)$ ne coïncide pas avec le processus initialement étudié par RÜSCHENDORF (1976) et repris dans la section 5.2 de RÉMILLARD (2010) ainsi que dans la section 3.2 de BÜCHER et RUPPERT (2013). Ce dernier est défini par

$$\mathbb{C}_n^R(s,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} \left\{ \mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n} \leq \boldsymbol{u}) - C(\boldsymbol{u}) \right\}, \qquad (s,\boldsymbol{u}) \in [0,1]^{d+1}. \tag{2.15}$$

Contrairement à $\mathbb{C}_n$, ce processus ne peut s'écrire en terme de la copule empirique que pour $s = 1$. En effet, les pseudo-observations utilisées, à savoir $\hat{\boldsymbol{U}}_1^{1:n}, \dots, \hat{\boldsymbol{U}}_n^{1:n}$, sont dans ce cas-ci évaluées une seule fois, à partir de l'échantillon complet et non sur les sous-échantillons $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$, pour $k \in \{1, \ldots, n\}$.

De façon similaire à la proposition 2.2.1, nous verrons que la loi limite du processus $\mathbb{C}_n$ dans $\ell^\infty(\Delta \times [0,1]^d)$ fait intervenir la loi limite du processus

$$\mathbb{Z}_n(s, \boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} \{\mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}) - C(\boldsymbol{u})\}, \qquad (s, \boldsymbol{u}) \in [0,1]^{d+1}, \tag{2.16}$$

avec la convention que $\mathbb{Z}_n(s, .) = 0$ pour tout $s < 1/n$.

Sous les conditions de la proposition 2.2.2, le processus $\mathbb{Z}_n$ converge dans $\ell^\infty([0,1]^{d+1})$ vers un processus de Kiefer-Müller $\mathbb{Z}_C$, c'est-à-dire un processus Gaussien tendu, centré, dont la structure de covariance est donnée pour $(s, \boldsymbol{u}), (t, \boldsymbol{v}) \in [0,1]^{d+1}$ par

$$\operatorname{cov}\{\mathbb{Z}_C(s, \boldsymbol{u}), \mathbb{Z}_C(t, \boldsymbol{v})\} = \min(s,t) \sum_{k \in \mathbb{Z}} \operatorname{cov}\{\mathbf{1}(\boldsymbol{U}_0 \leq \boldsymbol{u}), \mathbf{1}(\boldsymbol{U}_k \leq \boldsymbol{v})\}. \tag{2.17}$$

À noter que dans le cas d'observations i.i.d., la convergence en loi de $\mathbb{Z}_n$ vers $\mathbb{Z}_C$ est une conséquence directe du théorème 2.12.1 van der Vaart et Wellner (2000). Dans ce cas particulier, la covariance du processus $\mathbb{Z}_C$ se simplifie comme suit :

$$\operatorname{cov}\{\mathbb{Z}_C(s, \boldsymbol{u}), \mathbb{Z}_C(t, \boldsymbol{v})\} = \min(s,t)\{C(\boldsymbol{u} \wedge \boldsymbol{v}) - C(\boldsymbol{u})C(\boldsymbol{v})\}. \tag{2.18}$$

Il est également possible de définir la version doublement séquentielle du processus empirique $\mathbb{Z}_n$ défini dans (2.16) :

$$\begin{aligned} \mathbb{B}_n(s, t, \boldsymbol{u}) &= \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \{\mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}) - C(\boldsymbol{u})\} \\ &= \mathbb{Z}_n(t, \boldsymbol{u}) - \mathbb{Z}_n(s, \boldsymbol{u}), \qquad (s, t, \boldsymbol{u}) \in \Delta \times [0,1]^d, \end{aligned} \tag{2.19}$$

avec la convention que $\mathbb{B}_n(s, t, .) = 0$ dès que $\lfloor nt \rfloor - \lfloor ns \rfloor = 0$.

La convergence faible du processus $\mathbb{Z}_n$ dans $\ell^\infty([0,1]^{d+1})$ et le théorème des applications continues permettent d'obtenir que le processus doublement séquentiel $\mathbb{B}_n$ converge faiblement dans l'espace $\ell^\infty(\Delta \times [0,1]^d)$ vers le processus $\mathbb{B}_C$ défini par

$$\mathbb{B}_C(s, t, \boldsymbol{u}) = \mathbb{Z}_C(t, \boldsymbol{u}) - \mathbb{Z}_C(s, \boldsymbol{u}), \qquad (s, t, \boldsymbol{u}) \in \Delta \times [0,1]^d. \tag{2.20}$$

La proposition suivante est un corollaire du théorème 1 de Bücher et Kojadinovic (2013) et du théorème 1 de Bücher (2013). Il établit la convergence faible du processus $\mathbb{C}_n$ dans $\ell^\infty(\Delta \times [0,1]^d)$.

Proposition 2.3.1. *Supposons que $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ soient issus d'une suite $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$ strictement stationnaire, dont les marges sont continues et dont le coefficient de mélange satisfait $\alpha_r = O(r^{-a})$, $a > 1$. Alors, sous les conditions 2.1.1 et 2.2.1,*

$$\sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left|\mathbb{C}_n(s,t,\boldsymbol{u}) - \widetilde{\mathbb{C}}_n(s,t,\boldsymbol{u})\right| \xrightarrow{\mathrm{P}} 0, \tag{2.21}$$

avec pour $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$

$$\widetilde{\mathbb{C}}_n(s,t,\boldsymbol{u}) = \mathbb{B}_n(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u})\,\mathbb{B}_n(s,t,\boldsymbol{u}^{\{j\}}). \tag{2.22}$$

Par conséquent, $\mathbb{C}_n \rightsquigarrow \mathbb{C}_C$ dans $\ell^\infty(\Delta\times[0,1]^d)$, où pour $(s,t,\boldsymbol{u}) \in \Delta\times[0,1]^d$,

$$\mathbb{C}_C(s,t,\boldsymbol{u}) = \mathbb{B}_C(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u})\,\mathbb{B}_C(s,t,\boldsymbol{u}^{\{j\}}). \tag{2.23}$$

Dans l'article de la section 3.2, nous verrons que la condition 2.1.1 n'est pas nécessaire pour montrer la convergence (2.21).

2.3.2 rééchantillonnage à base de multiplicateurs dans le cas séquentiel et pour des données fortement mélangeantes

Pour un grand entier M, soient à présent M copies indépendantes $(\xi_i^{(1)})_{i\in\mathbb{Z}}, \dots, (\xi_i^{(M)})_{i\in\mathbb{Z}}$ d'une même suite i.i.d. de multiplicateurs, et considérons, pour $m \in \{1,\dots,M\}$, les processus empiriques séquentiels non évaluables,

$$\mathbb{Z}_n^{(m)}(s,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_i^{(m)} \{\mathbf{1}(\boldsymbol{U}_i \le \boldsymbol{u}) - C(\boldsymbol{u})\}, \qquad (s,\boldsymbol{u}) \in [0,1]^{d+1}.$$

Dans leur théorème 1, HOLMES et coll. (2013) étendent partiellement le théorème de la limite centrale pour multiplicateurs de KOSOROK (2008, théorème 10.1 et corollaire 10.3) au cas séquentiel. Comme corollaire de ce résultat, on a :

$$\left(\mathbb{Z}_n, \mathbb{Z}_n^{(1)}, \dots, \mathbb{Z}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{Z}_C, \mathbb{Z}_C^{(1)}, \dots, \mathbb{Z}_C^{(M)}\right)$$

dans $\{\ell^\infty([0,1]^{d+1})\}^{M+1}$, où $\mathbb{Z}_C$ est un processus Kiefer-Müller et les processus $\mathbb{Z}_C^{(1)}, \dots, \mathbb{Z}_C^{(M)}$ sont des copies indépendantes du processus $\mathbb{Z}_C$. Comme précédemment, lorsque n est grand, les processus $\mathbb{Z}_n^{(1)}, \dots, \mathbb{Z}_n^{(M)}$ peuvent être vus comme des copies «presque» indépendantes du processus $\mathbb{Z}_n$.

Pour $m \in \{1, \ldots, M\}$, définissons à présent les processus doublement séquentiels

$$\begin{aligned} \mathbb{B}_n^{(m)}(s, t, \boldsymbol{u}) &= \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \xi_i^{(m)} \{\mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}) - C(\boldsymbol{u})\} \\ &= \mathbb{Z}_n^{(m)}(t, \boldsymbol{u}) - \mathbb{Z}_n^{(m)}(s, \boldsymbol{u}), \qquad (s, t, \boldsymbol{u}) \in \Delta \times [0,1]^d, \end{aligned} \tag{2.24}$$

avec la convention dans la première écriture que $\mathbb{B}_n^{(m)}(s, t, .) = 0$ dès que $\lfloor ns \rfloor = \lfloor nt \rfloor$. Le théorème des applications continues implique alors que

$$\left(\mathbb{B}_n, \mathbb{B}_n^{(1)}, \ldots, \mathbb{B}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{B}_C, \mathbb{B}_C^{(1)}, \ldots, \mathbb{B}_C^{(M)}\right), \tag{2.25}$$

dans l'espace $\{\ell^\infty(\Delta \times [0,1]^d)\}^{M+1}$, où $\mathbb{B}_C$ est la loi limite du processus $\mathbb{B}_n$ dans $\ell^\infty(\Delta \times [0,1]^d)$ donnée dans (2.20), et les processus $\mathbb{B}_C^{(1)}, \ldots, \mathbb{B}_C^{(M)}$ sont des copies indépendantes du processus $\mathbb{B}_C$. Les processus $\mathbb{B}_n^{(1)}, \ldots, \mathbb{B}_n^{(M)}$ peuvent être vus, pour n grand, comme des copies «presque» indépendantes du processus $\mathbb{B}_n$.

Le résultat donné dans (2.25) est le point de départ, par analogie à la section 2.2.2, d'un rééchantillonnage à base de multiplicateurs pour le processus $\mathbb{C}_n$, défini dans (2.14), dans le cas de données sériellement dépendantes.

Lorsque les vecteurs $\boldsymbol{X}_1, \ldots \boldsymbol{X}_n$ sont sériellement dépendants, comme alternative au rééchantillonnage de Fermanian et coll. (2004), il est possible d'utiliser des techniques de rééchantillonnage par bloc qu'on pourra trouver dans Künsch (1989) et Bühlmann (1993) qui permettent de tenir compte de la dépendance sérielle des données.

La méthode des multiplicateurs i.i.d. peut être étendue au cas de donnés sériellement dépendantes. En partant des résultats de la section 3.3 de Bühlmann (1993), Bücher et Ruppert (2013) et Bücher et Kojadinovic (2013) ont étendu les techniques de rééchantillonnage de Rémillard et Scaillet (2009) au cas séquentiel et au cas de données fortement mélangeantes. L'idée de Bühlmann (1993) est de remplacer les multiplicateurs i.i.d. par des multiplicateurs sériellement dépendants, ce qui va permettre de la même façon que le rééchantillonnage par bloc, de conserver la structure de dépendance sérielle des données.

Définition 2.3.1. *Une suite de multiplicateurs $(\xi_{i,n})_{i \in \mathbb{Z}}$ est appelée suite dépendantes de multiplicateurs si elle est strictement stationnaire, indépendantes des observations $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ et vérifie les conditions (***M1***) – (***M3***) ci-après :*

(M1) $\mathbb{E}_\xi(\xi_{0,n}) = 0$, $Var_\xi(\xi_{0,n}) = 1$ *et* $\sup_{n\geq 1} \mathbb{E}\left(|\xi_{0,n}|^\nu\right) < \infty$ *quel que soit* $\nu \geq 1$.

(M2) *Il existe une suite d'entiers positifs* $\ell_n \to \infty$ *vérifiant* $\ell_n = o(n)$, *telle que la suite* $(\xi_{i,n})_{i\in\mathbb{Z}}$ *est* ℓ_n*-dépendante ; c'est-à-dire, telle que* $\xi_{i,n}$ *est indépendant de* $\xi_{i+h,n}$ *pour tout* $h > \ell_n$ *et tout* $i \in \mathbb{N}$.

(M3) *Il existe une fonction* $\varphi : \mathbb{R} \to [0,1]$, *symétrique et continue en 0, satisfaisant* $\varphi(0) = 1$ *et* $\varphi(x) = 0$ *pour tout* $|x| > 1$, *telle que* $\mathbb{E}(\xi_{0,n}\xi_{h,n}) = \varphi(h/\ell_n)$ *pour tout* $h \in \mathbb{Z}$.

Pour un grand entier M, soient $(\xi_{i,n}^{(1)})_{i\in\mathbb{Z}}, \ldots, (\xi_{i,n}^{(M)})_{i\in\mathbb{Z}}$ des copies indépendantes d'une même suite dépendante de multiplicateurs. Définissons alors les processus

$$\tilde{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi_{i,n}^{(m)} \{\mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}) - C(\boldsymbol{u})\}, \quad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d,$$

pour $m \in \{1,\ldots,M\}$. On a alors le résultat suivant, qui est le théorème 2.1 de BÜCHER et KOJADINOVIC (2013) :

Théorème 2.3.1. *Supposons que* $\ell_n = O(n^{1/2-\gamma})$ *avec* $0 < \gamma < 1/2$ *et que* $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$ *soient issus d'une suite* $(\boldsymbol{U}_i)_{i\in\mathbb{Z}}$ *strictement stationnaire dont le coefficient de mélange satisfait* $\alpha_r = O(r^{-a})$, $a > 3 + 3d/2$. *Alors*

$$\left(\mathbb{B}_n, \tilde{\mathbb{B}}_n^{(1)}, \ldots, \tilde{\mathbb{B}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{B}_C, \mathbb{B}_C^{(1)}, \ldots, \mathbb{B}_C^{(M)}\right), \quad \textit{dans } \{\ell^\infty(\Delta \times [0,1]^d)\}^{M+1},$$

où $\mathbb{B}_n$ *et* $\mathbb{B}_C$ *sont respectivement définis dans* (2.19) *et* (2.20) *et* $\mathbb{B}_C^{(1)}, \ldots, \mathbb{B}_C^{(M)}$ *sont des copies indépendantes du processus* $\mathbb{B}_C$.

Les processus $\tilde{\mathbb{B}}_n^{(m)}$ pour $m \in \{1,\ldots,M\}$ ne sont pas calculables, puisque les vecteurs $\boldsymbol{U}_i$ pour $i \in \{1,\ldots,n\}$ sont inconnus ainsi que la copule C. Il va donc être nécessaire d'estimer ces processus. Dans BÜCHER et KOJADINOVIC (2013), les processus $\hat{\mathbb{B}}_n^{(m)}$ ci-après sont proposés :

$$\hat{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi_{i,n}^{(m)} \{\mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n} \leq \boldsymbol{u}) - C_{1:n}(\boldsymbol{u})\}.$$

Ces derniers permettent de construire les processus calculables suivants :

$$\hat{\mathbb{C}}_n^{(m)}(s,t,\boldsymbol{u}) = \hat{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_{j,1:n}(\boldsymbol{u})\, \hat{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}^{\{j\}}), \quad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d,$$

où $\dot{C}_{j,1:n}$ satisfait la condition 2.2.2. On a alors la proposition suivante (proposition 4.2 de BÜCHER et KOJADINOVIC, 2013) :

Proposition 2.3.2. *Supposons que* $\ell_n = O(n^{1/2-\gamma})$ *où* $0 < \gamma < 1/2$ *et que* $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ *sont issus d'une suite* $(\boldsymbol{X}_i)_{i \in \mathbb{Z}}$ *strictement stationnaire dont le coefficient de mélange satisfait* $\alpha_r = O(r^{-a})$, $a > 3 + 3d/2$. *Alors sous les conditions 2.1.1, 2.2.1 et 2.2.2, on a :*

$$\left(\mathbb{C}_n, \hat{\mathbb{C}}_n^{(1)}, \ldots, \hat{\mathbb{C}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{C}_C, \mathbb{C}_C^{(1)}, \ldots, \mathbb{C}_C^{(M)}\right),$$

dans $\{\ell^\infty(\Delta \times [0,1]^d)\}^{M+1}$, *où* $\mathbb{C}_C^{(1)}, \ldots, \mathbb{C}_C^{(M)}$ *sont des copies indépendantes du processus* $\mathbb{C}_C$.

Bien évidemment, une suite de variables aléatoires i.i.d. est fortement mélangeante. Ses coefficients de mélange α_r sont nuls quel que soit $r > 0$. Ainsi, dans le cas de vecteurs aléatoires $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ i.i.d., les suites de multiplicateurs choisies pourront être des suites i.i.d., satisfaisants (**M**0), ou bien des suites dépendantes, satisfaisants (**M**1) – (**M**3). Néanmoins, en ce qui concerne leurs utilisations dans le domaine des tests non paramétriques que nous abordons dans la partie 2, les simulations révèlent que l'utilisation de suite i.i.d. de multiplicateurs dans le cas de données i.i.d. conduit à des tests plus performants à taille d'échantillon finie (voir Bücher et Ruppert, 2013) qu'en utilisant des suites dépendantes de multiplicateurs, ce qui est conforme à l'intuition. Lorsque l'hypothèse d'indépendance des données est satisfaite, il est donc préférable d'utiliser des multiplicateurs basés sur (**M**0) dans la construction des tests statistiques.

Comme alternative au rééchantillonnage du processus $\mathbb{C}_n$ proposé par Bücher et Kojadinovic (2013), il semble intéressant de considérer un rééchantillonnage de $\mathbb{C}_n$, construit de façon analogue, mais fondé sur les processus $\check{\mathbb{B}}_n^{(m)}$ ci-après :

$$\check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \xi_{i,n}^{(m)} \{\mathbf{1}(\hat{\boldsymbol{U}}_i^{\lfloor ns \rfloor+1:\lfloor nt \rfloor} \leq \boldsymbol{u}) - C_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u})\},$$

$(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$ et $m \in \{1, \ldots, M\}$. La validité asymptotique de cette méthode de rééchantillonnage est l'une des contribution de cette thèse et sera démontrée dans l'article de la section 3.2.

Deuxième partie

Tests de détection de rupture dans la dépendance multivariée

Chapitre 3

Test à la Cramér-von Mises, fondé sur le processus de copule empirique séquentiel

Sommaire

3.1 Introduction

Considérons une suite $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ de vecteurs aléatoires de dimension d, dont les marges sont supposées continues. Nous cherchons à mettre en place des procédures non paramétriques pour tester l'hypothèse suivante :

$$\mathcal{H}_0 : \qquad \exists F \text{ telle que } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n \text{ admettent } F \text{ pour fonction de répartition.} \tag{3.1}$$

Dans ce chapitre, nous allons adapter l'approche de Csörgő et Horváth (1997, Section 2.6) afin d'obtenir un test particulièrement sensible à des changements dans la copule d'observations multivariées fortement mélangeantes. Avant d'exposer les détails de notre approche, nous allons rappeler les grandes lignes de l'approche des auteurs mentionnés ci-dessus. Cette dernière est fondée sur la statistique suivante :

$$\begin{aligned} T_n^{\#} &= \max_{1 \le k \le n-1} \frac{k(n-k)}{n^{3/2}} \sup_{\boldsymbol{x} \in \mathbb{R}^d} |F_{1:k}(\boldsymbol{x}) - F_{k+1:n}(\boldsymbol{x})| \\ &= \sup_{s \in [0,1]} \sup_{x \in \mathbb{R}^d} \left| \mathbb{D}_n^{\#}(s, \boldsymbol{x}) \right|, \end{aligned}$$

avec, pour $(s, \boldsymbol{x}) \in [0,1] \times \mathbb{R}^d$,

$$\begin{aligned} \mathbb{D}_n^{\#}(s, \boldsymbol{x}) &= \sqrt{n} \lambda_n(0,s) \lambda_n(s,1) \{ F_{1:\lfloor ns \rfloor}(\boldsymbol{x}) - F_{\lfloor ns \rfloor + 1:n}(\boldsymbol{x}) \} && (3.2) \\ &= \sqrt{n} \lambda_n(0,s) \{ F_{1:\lfloor ns \rfloor}(\boldsymbol{x}) - F_{1:n}(\boldsymbol{x}) \}, && (3.3) \end{aligned}$$

où pour $s \leq t \in [0,1]$, $\lambda_n(s,t) = (\lfloor nt \rfloor - \lfloor ns \rfloor)/n$ et où pour $1 \leq k \leq l \leq n$, $F_{k:l}$ désigne la f.d.r. empirique de dimension d calculée sur le sous-ensemble $\boldsymbol{X}_k, \dots, \boldsymbol{X}_l$:

$$F_{k:l}(\boldsymbol{x}) = \frac{1}{l-k+1} \sum_{i=k}^{l} \mathbf{1}(\boldsymbol{X}_i \leq \boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^d.$$

L'expression (3.3) est la forme CUSUM du processus empirique de test. Plusieurs procédures non paramétriques pour tester $\mathcal{H}_0$ sont fondées sur la forme CUSUM mentionnée ci-dessus. On peut citer entre autres BAI (1994), GOMBAY et HORVÁTH (1999), INOUE (2001), ou encore de façon plus récente HOLMES et coll. (2013). Lorsque l'hypothèse $\mathcal{H}_0$ est vérifiée, les f.d.r. empiriques $F_{1:k}$ et $F_{k+1:n}$ des sous échantillons $\boldsymbol{X}_1, \dots, \boldsymbol{X}_k$ et $\boldsymbol{X}_{k+1}, \dots, \boldsymbol{X}_n$ pour k variant de 1 à $n-1$ estimeront la même f.d.r. F, ce qui implique que la statistique $T_n^{\#}$ devrait être petite. En revanche, on peut se convaincre que $T_n^{\#}$ devrait être plus grande dans le cas d'une rupture abrupte dans la f.d.r. des vecteurs $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$, c'est-à-dire, lorsque $\mathcal{H}_1$ est vraie, où $\mathcal{H}_1$ est définie par :

$\mathcal{H}_1$: Il existe $k^\star \in \{1, \dots, n-1\}$ et deux f.d.r. $F_1 \neq F_2$, telles que, F_1 est la f.d.r. des vecteurs $\boldsymbol{X}_1, \dots, \boldsymbol{X}_{k^\star}$ et F_2 est la f.d.r. des vecteurs $\boldsymbol{X}_{k^\star+1}, \dots, \boldsymbol{X}_n$.

L'approche proposée dans l'article ci-après consiste essentiellement à remplacer les f.d.r. empiriques dans la construction précédente par des copules empiriques. L'analogue du processus $\mathbb{D}_n^{\#}$ défini dans (3.2) est alors donné par :

$$\mathbb{D}_n(s, \boldsymbol{u}) = \sqrt{n}\lambda_n(0,s)\lambda_n(s,1)\{C_{1:\lfloor ns \rfloor}(\boldsymbol{u}) - C_{\lfloor ns \rfloor+1:n}(\boldsymbol{u})\}, \quad (s, \boldsymbol{u}) \in [0,1]^{d+1}, \tag{3.4}$$

où $C_{1:\lfloor ns \rfloor}$ et $C_{\lfloor ns \rfloor+1:n}$ sont définis dans (2.10). Dans l'article ci-après, nous nous intéressons particulièrement à des statistiques à la Cramér-von Mises construites à partir du processus $\mathbb{D}_n$.

Pour mieux comprendre l'intérêt du processus $\mathbb{D}_n$ dans le cas d'une suite d'observations dont les marges sont continues, remarquons que le théorème de Sklar (énoncé dans le chapitre 1) nous permet de réécrire l'hypothèse $\mathcal{H}_0$ comme $\mathcal{H}_{0,m} \cap \mathcal{H}_{0,c}$, où

$\mathcal{H}_{0,c}$: $\exists C$ telle que C est la copule des vecteurs aléatoires $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$,

$\mathcal{H}_{0,m}$: $\exists F_1, \dots, F_d$ telles que $F_1, \dots, F_d$ sont les f.d.r. marginales de $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$.

Les tests basés sur $\mathbb{D}_n^{\#}$ défini dans (3.2), s'avèrent très peu puissants pour des échantillons de taille modérée, lorsqu'on a à faire à une alternative de la forme $\mathcal{H}_{0,m} \cap \neg(\mathcal{H}_{0,c})$. Cette constatation est par exemple illustrée dans les simulations effectuées dans Holmes et coll. (2013, section 4). En partant de (3.4), on peut aisément voir que, par construction, les tests fondés sur le processus $\mathbb{D}_n$ devraient, quant à eux, être particulièrement sensibles à l'alternative $\mathcal{H}_{0,m} \cap (\neg\mathcal{H}_{0,c})$. En revanche, il est important de remarquer que ces procédures statistiques ne permettront pas de tester directement l'hypothèse $\mathcal{H}_{0,c}$. C'est seulement si l'hypothèse $\mathcal{H}_{0,m}$ est vraie que le rejet de $\mathcal{H}_0$, définie dans (3.1), permettra de conclure au rejet de $\mathcal{H}_{0,c}$.

Detecting changes in cross-sectional dependence in multivariate time series

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Abstract

Classical and more recent tests for detecting distributional changes in multivariate time series often lack power against alternatives that involve changes in the cross-sectional dependence structure. To be able to detect such changes better, a test is introduced based on a recently studied variant of the sequential empirical copula process. In contrast to earlier attempts, ranks are computed with respect to relevant subsamples, with beneficial consequences for the sensitivity of the test. For the computation of p-values we propose a multiplier resampling scheme that takes the serial dependence into account. The large-sample theory for the test statistic and the resampling scheme is developed. The finite-sample performance of the procedure is assessed by Monte Carlo simulations. Two case studies involving time series of financial returns are presented as well.

1 Introduction

Given a sequence $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ of d-dimensional observations, change-point detection aims at testing

$$H_0 : \exists\, F \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have c.d.f. } F \tag{1.1}$$

against alternatives involving the nonconstancy of the c.d.f. Under H_0 and the assumption that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ have continuous marginal c.d.f.s $F_1, \ldots, F_d$, we have from the work of Sklar (1959) that the common multivariate c.d.f. F can be written in a unique way as

$$F(\boldsymbol{x}) = C\{F_1(x_1), \ldots, F_d(x_d)\}, \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

where the function $C : [0,1]^d \to [0,1]$ is a *copula* and can be regarded as capturing the dependence between the components of $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$. It follows that H_0 can be rewritten as $H_{0,m} \cap H_{0,c}$, where

$$H_{0,m} : \exists\, F_1, \ldots, F_d \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have marginal c.d.f.s } F_1, \ldots, F_d, \tag{1.2}$$

$$H_{0,c} : \exists\, C \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have copula } C. \tag{1.3}$$

Classical nonparametric tests for H_0 are based on sequential empirical processes; see e.g. Bai (1994), Csörgő and Horváth (1997, Section 2.6) and Inoue (2001). For moderate sample sizes, however, such tests appear to have little power against alternative hypotheses that leave the margins unchanged but that involve a change in the copula, i.e., when $H_{0,m} \cap (\neg H_{0,c})$ holds. Empirical evidence of the latter fact can be found in Holmes et al. (2013, Section 4). For that reason, nonparametric tests for change-point detection particularly sensitive to changes in the dependence structure are of practical interest.

Several tests designed to capture changes in cross-sectional dependence structure were proposed in the literature. Tests based on Kendall's tau were investigated by Gombay and Horváth (1999) (see also Gombay and Horváth, 2002), Quessy et al. (2013) and Dehling et al. (2013). Although these have good power when the copula changes in such a way that Kendall's tau changes as well, they are obviously useless when the copula changes but Kendall's tau does not change or only very little. Tests based on functionals of sequential empirical copula processes were considered in Rémillard (2010), Bücher and Ruppert (2013), van Kampen and Wied (2013) and Wied et al. (2013). However, the power of such tests is often disappointing; see Section 5 for some numerical evidence.

It is our aim to construct a new test for H_0 that is more powerful than its predecessors against alternatives that involve a change in the copula. The test is based on sequential empirical copula processes as well, but the crucial difference lies in the computation of the ranks. Whereas in Rémillard (2010) and subsequent papers, ranks are always computed with respect to the full sample, we propose to compute the ranks with respect to the relevant subsamples; see Section 2 for details. The intuition is that in this way, the copulas of those subsamples are estimated more accurately, so that differences between copulas of disjoint subsamples are detected more quickly. The phenomenon is akin to the one observed in Genest and Segers (2010) that the empirical copula, which is based on pseudo-observations, is often a better estimator of a copula than the empirical distribution function based on observations from the copula itself. For another illustration in the context of tail dependence functions, see Bücher (2013a).

The paper is organized as follows. The test statistic is presented in Section 2, and its asymptotic distribution under the null hypothesis is found in Section 3. Next, Section 4 contains a detailed description of the multiplier resampling scheme and its asymptotic validity under the null hypothesis. The results of a large-scale Monte Carlo simulation study are reported in Section 5, and two brief case studies are given in Section 6. Section 7 concludes. Proofs and details regarding the simulation study are deferred to the Appendices.

In the rest of the paper, the arrow '$\rightsquigarrow$' denotes weak convergence in the sense of Definition 1.3.3 in van der Vaart and Wellner (2000). Given a set T, let $\ell^\infty(T)$ denote the space of all bounded real-valued functions on T equipped with the uniform metric.

2 Test statistic

We now describe our test statistic and highlight the difference with the one in Rémillard (2010) and Bücher and Ruppert (2013). Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be random vectors. For integers $1 \leq k \leq l \leq n$, let $C_{k:l}$ be the empirical copula of the sample $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$. Specifically,

$$C_{k:l}(\boldsymbol{u}) = \frac{1}{l-k+1} \sum_{i=k}^{l} \mathbf{1}(\hat{\boldsymbol{U}}_i^{k:l} \leq \boldsymbol{u}), \tag{2.1}$$

for $\boldsymbol{u} \in [0,1]^d$, where

$$\hat{\boldsymbol{U}}_i^{k:l} = \frac{1}{l-k+1}(R_{i1}^{k:l}, \ldots, R_{id}^{k:l}), \qquad i \in \{k, \ldots, l\}, \tag{2.2}$$

with $R_{ij}^{k:l} = \sum_{t=k}^{l} \mathbf{1}(X_{tj} \leq X_{ij})$ the (maximal) rank of X_{ij} among $X_{kj}, \ldots, X_{lj}$. (Because of serial dependence, there can be ties, even if the marginal distribution is continuous; think for instance of a moving maximum process.) An important point is that the ranks are computed within the subsample $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$ and not within the whole sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$. As we continue, we adopt the convention that $C_{k:l} = 0$ if $k > l$.

Write $\Delta = \{(s,t) \in [0,1]^2 : s \leq t\}$. Let $\lambda_n(s,t) = (\lfloor nt \rfloor - \lfloor ns \rfloor)/n$ for $(s,t) \in \Delta$. Our test statistic is based on the difference process, $\mathbb{D}_n$, defined by

$$\mathbb{D}_n(s, \boldsymbol{u}) = \sqrt{n}\, \lambda_n(0,s)\, \lambda_n(s,1)\, \{C_{1:\lfloor ns \rfloor}(\boldsymbol{u}) - C_{\lfloor ns \rfloor + 1:n}(\boldsymbol{u})\} \tag{2.3}$$

for $(s, \boldsymbol{u}) \in [0,1]^{d+1}$. For every $s \in [0,1]$, it gives a weighted difference between the empirical copulas at $\boldsymbol{u}$ of the first $\lfloor ns \rfloor$ and the last $n - \lfloor ns \rfloor$ points of the sample. Large absolute differences point in the direction of a change in the copula.

To aggregate over $\boldsymbol{u}$, we consider the Cramér–von Mises statistic

$$S_{n,k} = \int_{[0,1]^d} \{\mathbb{D}_n(k/n, \boldsymbol{u})\}^2 \, \mathrm{d}C_{1:n}(\boldsymbol{u}), \qquad k \in \{1, \ldots, n-1\}.$$

The test statistic for detecting changes in cross-sectional dependence is then

$$S_n = \max_{1\leq k\leq n-1} S_{n,k} = \sup_{s\in[0,1]} \int_{[0,1]^d} \{\mathbb{D}_n(s,\boldsymbol{u})\}^2 \,\mathrm{d}C_{1:n}(\boldsymbol{u}). \tag{2.4}$$

Other aggregating functions can be thought of too, leading for instance to Kolmogorov–Smirnov and Kuiper statistics. In numerical experiments, the resulting tests were found to be less powerful than the one based on the Cramér–von Mises statistic and hence are not considered further in this paper.

The null hypothesis of a constant distribution is rejected when S_n is large. The p-values are determined by the null distribution of S_n, whose large-sample limit is derived in Section 3. To estimate the p-values from the data, a multiplier bootstrap method is proposed in Section 4.

Finally, if H_0 is rejected, there could be one or several abrupt or smooth changes in the joint distribution. Moreover, the change(s) could concern one or more marginal distributions, the copula, or both. In the case where there is just a single (abrupt) change-point $k^* \in \{1,\ldots,n-1\}$, one can for instance estimate it by

$$k_n^\star = \operatorname*{arg\,max}_{1\leq k\leq n-1} S_{n,k}. \tag{2.5}$$

We do not pursue the issue of single or multiple change-point estimation nor the diagnosis of the nature of the change-point.

Our test statistic S_n differs from the one considered in Rémillard (2010, Section 5.2) and Bücher and Ruppert (2013, Section 3.2) in the way the copulas of the subsamples $\boldsymbol{X}_k,\ldots,\boldsymbol{X}_l$ are estimated. Rather than the empirical copula $C_{k:l}$, these authors propose to use

$$C_{k:l,n}(\boldsymbol{u}) = \frac{1}{l-k+1}\sum_{i=k}^{l} \mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n} \leq \boldsymbol{u}), \qquad \boldsymbol{u}\in[0,1]^d, \tag{2.6}$$

with the convention that $C_{k:l,n} = 0$ if $k > l$. In comparison with $C_{k:l}$ in (2.1), the ranks for the subsample $\boldsymbol{X}_k,\ldots,\boldsymbol{X}_l$ are computed relative to the complete sample $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$. The estimators $C_{k:l,n}$ yield the difference process

$$\mathbb{D}_n^R(s,\boldsymbol{u}) = \sqrt{n}\,\lambda_n(0,s)\,\lambda_n(s,1)\,\{C_{1:\lfloor ns\rfloor,n}(\boldsymbol{u}) - C_{\lfloor ns\rfloor+1:n,n}(\boldsymbol{u})\} \tag{2.7}$$

for $(s,\boldsymbol{u})\in[0,1]^{d+1}$. The process $\mathbb{D}_n^R$ is to be compared with the process $\mathbb{D}_n$ in (2.3). The difference lies in the use of $C_{k:l,n}$ rather than $C_{k:l}$. From the process $\mathbb{D}_n^R$, one obtains the test statistic

$$S_n^R = \sup_{s\in[0,1]} \int_{[0,1]^d} \left\{\mathbb{D}_n^R(s,\boldsymbol{u})\right\}^2 \mathrm{d}C_{1:n}(\boldsymbol{u}), \tag{2.8}$$

which is the analogue of S_n in (2.4).

In the Monte Carlo simulation experiments (Section 5), we will see that S_n is usually more powerful than S_n^R for detecting changes in the cross-sectional copula. Intuitively, the reason is that the empirical copula $C_{k:l}$ in (2.1) is often a better copula estimator than $C_{k:l,n}$ in (2.6). Note that $C_{k:l}$ is not only the empirical copula of $\boldsymbol{X}_k,\ldots,\boldsymbol{X}_l$, it is also equal to the empirical copula of $\hat{\boldsymbol{U}}_k^{1:n}, \ldots, \hat{\boldsymbol{U}}_l^{1:n}$, of which $C_{k:l,n}$ is the empirical distribution function.

In Genest and Segers (2010), situations are identified where the empirical copula of an independent random sample drawn from a given bivariate copula has a lower asymptotic variance than the empirical distribution function of that sample. Of course, the situation here is different from the one in the cited paper: multivariate rather than bivariate, serial dependence rather than independence. But still, we suspect the same mechanisms to be active.

3 Large-sample distribution

The asymptotic distribution under H_0 of our test statistic S_n in (2.4) can be obtained by writing it as a functional of the two-sided sequential empirical copula process studied in Bücher and Kojadinovic (2013). Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ be a strictly stationary d-variate time series with stationary c.d.f. F having continuous margins $F_1, \ldots, F_d$ and copula C. Recall $C_{k:l}$ in (2.1) and $\hat{\boldsymbol{U}}_i^{k:l}$ in (2.2). The two-sided sequential empirical copula process, $\mathbb{C}_n$, is defined by

$$\mathbb{C}_n(s,t,\boldsymbol{u}) = \sqrt{n}\,\lambda_n(s,t)\,\{C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - C(\boldsymbol{u})\} \tag{3.1}$$

$$= \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left\{\mathbf{1}(\hat{\boldsymbol{U}}_i^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \le \boldsymbol{u}) - C(\boldsymbol{u})\right\}, \tag{3.2}$$

for $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$. The link of $\mathbb{C}_n$ to our test statistic S_n in (2.4) is that, under H_0, the difference process $\mathbb{D}_n$ in (2.3) can be written as

$$\mathbb{D}_n(s,\boldsymbol{u}) = \lambda_n(s,1)\,\mathbb{C}_n(0,s,\boldsymbol{u}) - \lambda_n(0,s)\,\mathbb{C}_n(s,1,\boldsymbol{u}), \tag{3.3}$$

for $(s,\boldsymbol{u}) \in [0,1]^{d+1}$.

Before focusing on the weak limit of the process $\mathbb{D}_n$ under H_0, let us briefly recall the notion of *strongly mixing sequence*. For a sequence of d-dimensional random vectors $(\boldsymbol{Y}_i)_{i\in\mathbb{Z}}$, the σ-field generated by $(\boldsymbol{Y}_i)_{a\le i\le b}$, $a,b \in \mathbb{Z}\cup\{-\infty,+\infty\}$, is denoted by $\mathcal{F}_a^b$. The strong mixing coefficients corresponding to the sequence $(\boldsymbol{Y}_i)_{i\in\mathbb{Z}}$ are defined by

$$\alpha_r = \sup_{p\in\mathbb{Z}} \sup_{A\in\mathcal{F}_{-\infty}^p,\, B\in\mathcal{F}_{p+r}^{+\infty}} |P(A\cap B) - P(A)P(B)|$$

for positive integer r. The sequence $(\boldsymbol{Y}_i)_{i\in\mathbb{Z}}$ is said to be *strongly mixing* if $\alpha_r \to 0$ as $r \to \infty$.

The weak limit of the two-sided empirical copula process $\mathbb{C}_n$ defined in (3.2) under strong mixing was established in Bücher and Kojadinovic (2013) under the following two conditions:

Condition 3.1. *With probability one, there are no ties in each of the d component series $X_{1j}, X_{2j}, \ldots$, where $j \in \{1,\ldots,d\}$.*

Condition 3.2. *For any $j \in \{1,\dots,d\}$, the partial derivatives $\dot C_j = \partial C/\partial u_j$ exist and are continuous on $V_j = \{\boldsymbol{u} \in [0,1]^d : u_j \in (0,1)\}$.*

Condition 3.1 was considered in Bücher and Kojadinovic (2013) as continuity of the marginal distributions is *not* sufficient to guarantee the absence of ties when the observations are serially dependent (see e.g. Bücher and Segers, 2013, Example 4.2). One of the contributions of this work is to show that it actually can be dispensed with. Condition 3.2 was proposed in Segers (2012) and is nonrestrictive in the sense that it is necessary for the candidate weak limit of $\mathbb{C}_n$ to exist pointwise and have continuous trajectories.

As we continue, for any $j \in \{1,\dots,d\}$, we define $\dot C_j$ to be zero on the set $\{\boldsymbol{u} \in [0,1]^d : u_j \in \{0,1\}\}$ (see also Segers, 2012; Bücher and Volgushev, 2013). Also, for any $j \in \{1,\dots,d\}$ and any $\boldsymbol{u} \in [0,1]^d$, $\boldsymbol{u}^{(j)}$ is the vector of $[0,1]^d$ defined by $u_i^{(j)} = u_j$ if $i = j$ and 1 otherwise.

The weak convergence of the process $\mathbb{C}_n$ defined in (3.2) actually follows from that of the process

$$\mathbb{B}_n(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \{\mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}) - C(\boldsymbol{u})\}, \qquad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d, \tag{3.4}$$

where $\boldsymbol{U}_1,\dots,\boldsymbol{U}_n$ is the unobservable sample obtained from $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n$ by the probability integral transforms $U_{ij} = F_j(X_{ij})$, $i \in \{1,\dots,n\}$, $j \in \{1,\dots,d\}$, and with the convention that $\mathbb{B}_n(s,t,\cdot) = 0$ if $\lfloor nt\rfloor - \lfloor ns\rfloor = 0$.

If $\boldsymbol{U}_1,\dots,\boldsymbol{U}_n$ is drawn from a strictly stationary sequence $(\boldsymbol{U}_i)_{i\in\mathbb{Z}}$ whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$ with $a > 1$, we have from Bücher (2013b) that $\mathbb{B}_n(0,\cdot,\cdot)$ converges weakly in $\ell^\infty([0,1]^{d+1})$ to a tight centered Gaussian process $\mathbb{Z}_C$ with covariance function

$$\operatorname{cov}\{\mathbb{Z}_C(s,\boldsymbol{u}), \mathbb{Z}_C(t,\boldsymbol{v})\} = \min(s,t) \sum_{k\in\mathbb{Z}} \operatorname{cov}\{\mathbf{1}(\boldsymbol{U}_0 \leq \boldsymbol{u}), \mathbf{1}(\boldsymbol{U}_k \leq \boldsymbol{v})\}.$$

The latter is actually a consequence of Lemma 2 in Bücher (2013b) stating that $\mathbb{B}_n(0,\cdot,\cdot)$ is asymptotically uniformly equicontinuous in probability, which in turn implies that $\mathbb{Z}_C$ has continuous trajectories with probability one. As a consequence of the continuous mapping theorem, $\mathbb{B}_n \rightsquigarrow \mathbb{B}_C$ in $\ell^\infty(\Delta \times [0,1]^d)$, where

$$\mathbb{B}_C(s,t,\boldsymbol{u}) = \mathbb{Z}_C(t,\boldsymbol{u}) - \mathbb{Z}_C(s,\boldsymbol{u}), \qquad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d. \tag{3.5}$$

The following result is a consequence of Theorem 3.4 of Bücher and Kojadinovic (2013) and the arguments used in the proof of Lemma A.2 of Bücher and Segers (2013). Its proof is given in Appendix A.

Proposition 3.3. *Let $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n$ be drawn from a strictly stationary sequence $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$ with continuous margins and whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$, $a > 1$. Then, provided Condition 3.2 holds,*

$$\sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left|\mathbb{C}_n(s,t,\boldsymbol{u}) - \tilde{\mathbb{C}}_n(s,t,\boldsymbol{u})\right| \xrightarrow{\mathrm{P}} 0, \tag{3.6}$$

where

$$\tilde{\mathbb{C}}_n(s,t,\boldsymbol{u}) = \mathbb{B}_n(s,t,\boldsymbol{u}) - \sum_{j=1}^d \dot{C}_j(\boldsymbol{u})\,\mathbb{B}_n(s,t,\boldsymbol{u}^{(j)}). \tag{3.7}$$

Consequently, $\mathbb{C}_n \rightsquigarrow \mathbb{C}_C$ *in* $\ell^\infty(\Delta \times [0,1]^d)$*, where, for* $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$*,*

$$\mathbb{C}_C(s,t,\boldsymbol{u}) = \mathbb{B}_C(s,t,\boldsymbol{u}) - \sum_{j=1}^d \dot{C}_j(\boldsymbol{u})\,\mathbb{B}_C(s,t,\boldsymbol{u}^{(j)}). \tag{3.8}$$

In view of (3.3), the weak limit of $\mathbb{D}_n$ under H_0 is a mere corollary of Proposition 3.3 and the continuous mapping theorem.

Corollary 3.4. *Under the conditions of Proposition 3.3,* $\mathbb{D}_n \rightsquigarrow \mathbb{D}_C$ *in* $\ell^\infty([0,1]^{d+1})$*, where, for any* $(s,\boldsymbol{u}) \in [0,1]^{d+1}$*,*

$$\mathbb{D}_C(s,\boldsymbol{u}) = \mathbb{C}_C(0,s,\boldsymbol{u}) - s\,\mathbb{C}_C(0,1,\boldsymbol{u}), \tag{3.9}$$

with $\mathbb{C}_C$ *defined in* (3.8)*. As a consequence,*

$$S_n \rightsquigarrow S = \sup_{s\in[0,1]} \int_{[0,1]^d} \{\mathbb{D}_C(s,\boldsymbol{u})\}^2\,\mathrm{d}C(\boldsymbol{u}). \tag{3.10}$$

The covariance function of $\mathbb{D}_C$ can be expressed in terms of the one of $\mathbb{C}_C$ by

$$\mathrm{cov}\{\mathbb{D}_C(s,\boldsymbol{u}),\mathbb{D}_C(t,\boldsymbol{v})\} = \{\min(s,t) - st\}\ \mathrm{cov}\{\mathbb{C}_C(0,1,\boldsymbol{u}),\mathbb{C}_C(0,1,\boldsymbol{v})\}.$$

4 Resampling

In order to compute p-values for S_n based on (3.10), we propose to use resampling methods. Tracing back the definition of S_n via $\mathbb{D}_n$ to $\mathbb{C}_n$ in (3.1) and using the approximation via $\tilde{\mathbb{C}}_n$ in (3.7), we find that it suffices to construct a resampling scheme for $\mathbb{B}_n$ defined in (3.4) and to estimate the first-order partial derivatives, $\dot{C}_j$, of C.

4.1 Multiplier sequences

In the case of i.i.d. observations, Scaillet (2005) proposed to use a *multiplier* approach in the spirit of van der Vaart and Wellner (2000, Chapter 2.9) to resample $\mathbb{B}_n$. When the first-order partial derivatives of C are estimated by finite-differencing as in Rémillard and Scaillet (2009), the resulting resampling scheme for $\mathbb{C}_n$ is frequently referred to as a *multiplier bootstrap*. In a nonsequential setting based on independent observations, Bücher and Dette (2010) compared the finite-sample behavior of the various resampling techniques proposed in the literature and concluded that the multiplier bootstrap of Rémillard and Scaillet (2009) has, overall, the best finite-sample properties. This technique was revisited theoretically by Segers (2012) who showed its asymptotic validity under Condition 3.2. A sequential generalization of the latter result will be stated later in this section. In the case of independent observations, the multiplier bootstrap is based on *i.i.d. multiplier sequences*. We say that a sequence of random variables $(\xi_{i,n})_{i\in\mathbb{Z}}$ is an *i.i.d. multiplier sequence* if:

(M0) $(\xi_{i,n})_{i\in\mathbb{Z}}$ is i.i.d., independent of $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n$, with distribution not changing with n, having mean 0, variance 1, and being such that $\int_0^\infty \{\mathrm{P}(|\xi_{0,n}|>x)\}^{1/2}\mathrm{d}x<\infty$.

Starting from the seminal work of Bühlmann (1993, Section 3.3), Bücher and Ruppert (2013) and Bücher and Kojadinovic (2013) have studied a *dependent* multiplier bootstrap for $\mathbb{C}_n$ which extends the multiplier bootstrap of Rémillard and Scaillet (2009) to the sequential and strongly mixing setting. The key idea in Bühlmann (1993) is to replace i.i.d. multipliers by suitably serially dependent multipliers that will capture the serial dependence in the data. In the rest of the paper, we say that a sequence of random variables $(\xi_{i,n})_{i\in\mathbb{Z}}$ is a *dependent multiplier sequence* if:

(M1) The sequence $(\xi_{i,n})_{i\in\mathbb{Z}}$ is strictly stationary with $\mathrm{E}(\xi_{0,n})=0$, $\mathrm{E}(\xi_{0,n}^2)=1$ and $\sup_{n\geq 1}\mathrm{E}(|\xi_{0,n}|^\nu)<\infty$ for all $\nu\geq 1$, and is independent of the available sample $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n$.

(M2) There exists a sequence $\ell_n\to\infty$ of strictly positive constants such that $\ell_n=o(n)$ and the sequence $(\xi_{i,n})_{i\in\mathbb{Z}}$ is ℓ_n-dependent, i.e., $\xi_{i,n}$ is independent of $\xi_{i+h,n}$ for all $h>\ell_n$ and $i\in\mathbb{N}$.

(M3) There exists a function $\varphi:\mathbb{R}\to[0,1]$, symmetric around 0, continuous at 0, satisfying $\varphi(0)=1$ and $\varphi(x)=0$ for all $|x|>1$ such that $\mathrm{E}(\xi_{0,n}\xi_{h,n})=\varphi(h/\ell_n)$ for all $h\in\mathbb{Z}$.

Ways to generate dependent multiplier sequences are mentioned in Section 5 and Appendix C.

4.2 Computing p-values via resampling

Let M be a large integer and let $(\xi_{i,n}^{(1)})_{i\in\mathbb{Z}},\dots,(\xi_{i,n}^{(M)})_{i\in\mathbb{Z}}$ be M independent copies of the same multiplier sequence. We will define two multiplier resampling schemes for the process $\mathbb{B}_n$ in (3.4). These will lead to two resampling schemes for the test statistic S_n in (2.4), on the basis of which approximate p-values can be computed.

Recall that $C_{k:l}$ in (2.1) is the empirical copula of $\boldsymbol{X}_k,\dots,\boldsymbol{X}_l$, which is the empirical distribution of the vectors of rescaled ranks $\hat{\boldsymbol{U}}_i^{k:l}$ in (2.2). For any $m\in\{1,\dots,M\}$ and $(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d$, let

$$\hat{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u})=\frac{1}{\sqrt{n}}\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\xi_{i,n}^{(m)}\{\mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n}\leq\boldsymbol{u})-C_{1:n}(\boldsymbol{u})\},\tag{4.1}$$

and

$$\begin{aligned}\check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u})&=\frac{1}{\sqrt{n}}\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\xi_{i,n}^{(m)}\{\mathbf{1}(\hat{\boldsymbol{U}}_i^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}\leq\boldsymbol{u})-C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u})\}\\&=\frac{1}{\sqrt{n}}\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}(\xi_{i,n}^{(m)}-\bar{\xi}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{(m)})\mathbf{1}(\hat{\boldsymbol{U}}_i^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}\leq\boldsymbol{u}),\end{aligned}\tag{4.2}$$

where $\bar{\xi}^{(m)}_{k:l}$ is the arithmetic mean of $\xi^{(m)}_{i,n}$ for $i \in \{k, \ldots, l\}$. By convention, the sums are zero if $\lfloor ns \rfloor = \lfloor nt \rfloor$. Note that the ranks are computed relative to the complete sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ for $\hat{\mathbb{B}}^{(m)}_n(s,t,\cdot)$, whereas they are computed relative to the subsample $\boldsymbol{X}_{\lfloor ns \rfloor+1}, \ldots, \boldsymbol{X}_{\lfloor nt \rfloor}$ for $\check{\mathbb{B}}^{(m)}_n(s,t,\cdot)$.

In order to get to resampling versions of $\tilde{\mathbb{C}}_n$ in (3.7), we need estimators of the first-order partial derivatives of C. A simple estimator based on $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$ consists of finite differencing at a bandwidth of $h \equiv h(k,l) = \min\{(l-k+1)^{-1/2}, 1/2\}$. Varying slightly upon the definition in Rémillard and Scaillet (2009) and following Kojadinovic et al. (2011a, Section 3), we put

$$\dot{C}_{j,k:l}(\boldsymbol{u}) = \frac{C_{k:l}(\boldsymbol{u} + h\boldsymbol{e}_j) - C_{k:l}(\boldsymbol{u} - h\boldsymbol{e}_j)}{\min(u_j + h, 1) - \max(u_j - h, 0)}$$

for $\boldsymbol{u} \in [0,1]^d$, where $\boldsymbol{e}_j$ is the jth canonical unit vector in $\mathbb{R}^d$. Note that if $h \le u_j \le 1-h$, the denominator is just $2h$. The more general form of the denominator corrects for boundary effects (u_j close to 0 or 1). Proceeding for instance as in Kojadinovic et al. (2011a, proof of Proposition 2), we find that the previous estimator is uniformly bounded.

The resampling versions $\hat{\mathbb{B}}^{(m)}_n$ and $\check{\mathbb{B}}^{(m)}_n$ of $\mathbb{B}_n$ then lead to the following resampling versions for $\tilde{\mathbb{C}}_n$: for $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$,

$$\begin{aligned}
\hat{\mathbb{C}}^{(m)}_n(s,t,\boldsymbol{u}) &= \hat{\mathbb{B}}^{(m)}_n(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_{j,1:n}(\boldsymbol{u})\, \hat{\mathbb{B}}^{(m)}_n(s,t,\boldsymbol{u}^{(j)}), \\
\check{\mathbb{C}}^{(m)}_n(s,t,\boldsymbol{u}) &= \check{\mathbb{B}}^{(m)}_n(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_{j,\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u})\, \check{\mathbb{B}}^{(m)}_n(s,t,\boldsymbol{u}^{(j)}).
\end{aligned} \tag{4.3}$$

Recall that $\lambda_n(s,t) = (\lfloor nt \rfloor - \lfloor ns \rfloor)/n$. The difference process $\mathbb{D}_n$ is to be resampled by one of the following two methods:

$$\begin{aligned}
\hat{\mathbb{D}}^{(m)}_n(s,\boldsymbol{u}) &= \lambda_n(s,1)\, \hat{\mathbb{C}}^{(m)}_n(0,s,\boldsymbol{u}) - \lambda_n(0,s)\, \hat{\mathbb{C}}^{(m)}_n(s,1,\boldsymbol{u}) \\
&= \hat{\mathbb{C}}^{(m)}_n(0,s,\boldsymbol{u}) - \lambda_n(0,s)\, \hat{\mathbb{C}}^{(m)}_n(0,1,\boldsymbol{u}),
\end{aligned}$$

$$\check{\mathbb{D}}^{(m)}_n(s,\boldsymbol{u}) = \lambda_n(s,1)\, \check{\mathbb{C}}^{(m)}_n(0,s,\boldsymbol{u}) - \lambda_n(0,s)\, \check{\mathbb{C}}^{(m)}_n(s,1,\boldsymbol{u}).$$

For resampling the test statistic, one has the choice between

$$\hat{S}^{(m)}_n = \sup_{s \in [0,1]} \int_{[0,1]^d} \{\hat{\mathbb{D}}^{(m)}_n(s,\boldsymbol{u})\}^2 \, \mathrm{d}C_{1:n}(\boldsymbol{u}), \tag{4.4}$$

$$\check{S}^{(m)}_n = \sup_{s \in [0,1]} \int_{[0,1]^d} \{\check{\mathbb{D}}^{(m)}_n(s,\boldsymbol{u})\}^2 \, \mathrm{d}C_{1:n}(\boldsymbol{u}). \tag{4.5}$$

Finally, approximate p-values of the observed test statistic S_n can be computed via either

$$\frac{1}{M} \sum_{m=1}^{M} \mathbf{1}\left(\hat{S}^{(m)}_n \ge S_n\right) \qquad \text{or} \qquad \frac{1}{M} \sum_{m=1}^{M} \mathbf{1}\left(\check{S}^{(m)}_n \ge S_n\right). \tag{4.6}$$

The null hypothesis is rejected if the estimated p-value is smaller than the desired significance level.

By comparison, note that for the test statistic S_n^R in (2.8) based on the process $\mathbb{D}_n^R$ in (2.7), an approximate p-value can be computed using the multiplier processes

$$\mathbb{D}_n^{R,(m)}(s, \boldsymbol{u}) = \hat{\mathbb{B}}_n^{(m)}(0, s, \boldsymbol{u}) - \lambda_n(0, s)\, \hat{\mathbb{B}}_n^{(m)}(0, 1, \boldsymbol{u}), \tag{4.7}$$

where $\hat{\mathbb{B}}_n^{(m)}$ is defined in (4.1); see also Rémillard (2010, Section 5.2) and Bücher and Ruppert (2013, Section 3.2).

4.3 Asymptotic validity of the resampling scheme

We establish the asymptotic validity of the multiplier resampling schemes described above under the null hypothesis. First, we need to impose conditions on the data generating process $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ and the multiplier sequences $(\xi_{i,n}^{(m)})_{i \in \mathbb{Z}}$ for $m \in \{1, \dots, M\}$.

Condition 4.1. *One of the following two conditions holds:*

- **(i)** *The random vectors $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are i.i.d. and $(\xi_{i,n}^{(1)})_{i \in \mathbb{Z}}, \dots, (\xi_{i,n}^{(M)})_{i \in \mathbb{Z}}$ are independent copies of a multiplier sequence satisfying (M0).*
- **(ii)** *The random vectors $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are drawn from a strictly stationary sequence $(\boldsymbol{X}_i)_{i \in \mathbb{Z}}$ whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$ for some $a > 3 + 3d/2$, and $(\xi_{i,n}^{(1)})_{i \in \mathbb{Z}}, \dots, (\xi_{i,n}^{(M)})_{i \in \mathbb{Z}}$ are independent copies of a dependent multiplier sequence satisfying (M1)–(M3) with $\ell_n = O(n^{1/2-\gamma})$ for some $0 < \gamma < 1/2$.*

In both cases, the stationary distribution of $\boldsymbol{X}_i$ has continuous margins and a copula C satisfying Condition 3.2.

If the random vectors $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are i.i.d., they can also be considered to be drawn from a strongly mixing, strictly stationary sequence. Hence, for the multiplier sequences $(\xi_{i,n}^{(m)})_{i \in \mathbb{Z}}$, one could either assume (M0) or (M1)–(M3): both should work. However, as discussed in Bücher and Kojadinovic (2013, Section 2), the use of dependent multipliers in the case of independent observations is likely to result in an efficiency loss. This is illustrated in the Monte Carlo simulations reported in Bücher and Ruppert (2013, Section 3) and carried out for the test based on the statistic S_n^R defined in (2.8) which is resampled using multiplier processes asymptotically equivalent to those given in (4.7): the use of dependent multipliers in the case of serially independent data usually results in a loss of power and in a slightly more conservative test. Thus, in finite samples, if there is no evidence against serial independence, it appears more sensible to work under (M0).

We can now state the asymptotic distributions of the multiplier resampling schemes under the null hypothesis of a constant distribution. We provide two propositions, one for the resampling scheme based on $\hat{\mathbb{B}}_n^{(m)}$ in (4.1) and another one for the scheme based on $\check{\mathbb{B}}_n^{(m)}$ in (4.2).

Proposition 4.2. *If Condition 4.1 holds, then*

$$\left(\mathbb{C}_n, \hat{\mathbb{C}}_n^{(1)}, \dots, \hat{\mathbb{C}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{C}_C, \mathbb{C}_C^{(1)}, \dots, \mathbb{C}_C^{(M)}\right)$$

in $\{\ell^\infty(\Delta \times [0,1]^d)\}^{M+1}$*, where* $\mathbb{C}_C$ *is defined in* (3.8)*, and* $\mathbb{C}_C^{(1)}, \dots, \mathbb{C}_C^{(M)}$ *are independent copies of* $\mathbb{C}_C$*. As a consequence, also*

$$\left(\mathbb{D}_n, \hat{\mathbb{D}}_n^{(1)}, \dots, \hat{\mathbb{D}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{D}_C, \mathbb{D}_C^{(1)}, \dots, \mathbb{D}_C^{(M)}\right),$$

in $\{\ell^\infty([0,1]^{(d+1)})\}^{M+1}$*, where* $\mathbb{D}_C$ *is defined in* (3.9) *and* $\mathbb{D}_C^{(1)}, \dots, \mathbb{D}_C^{(M)}$ *are independent copies of* $\mathbb{D}_C$*. Finally,*

$$\left(S_n, \hat{S}_n^{(1)}, \dots, \hat{S}_n^{(M)}\right) \rightsquigarrow \left(S, S^{(1)}, \dots, S^{(M)}\right)$$

where S *is defined in* (3.10) *and* $S^{(1)}, \dots, S^{(M)}$ *are independent copies of* S*.*

Under Condition 4.1(i), the above result can be easily proved by starting from Theorem 1 of Holmes et al. (2013) and adapting the arguments used in Segers (2012, proof of Proposition 3.2). Under Conditions 4.1(ii) and 3.1, the result was obtained in Bücher and Kojadinovic (2013, Proposition 4.2). The additional arguments allowing to avoid Condition 3.1 will be given in the proof of the next result.

Proposition 4.3. *If Condition 4.1 holds, then the conclusions of Proposition 4.2 also hold with* $\hat{\mathbb{C}}_n^{(m)}$ *replaced by* $\check{\mathbb{C}}_n^{(m)}$*,* $\hat{\mathbb{D}}_n^{(m)}$ *replaced by* $\check{\mathbb{D}}_n^{(m)}$*, and* $\hat{S}_n^{(m)}$ *replaced by* $\check{S}_n^{(m)}$*.*

The proof of Proposition 4.3 is somewhat involved and is given in detail in Appendix B.

Combining the last claims of Propositions 4.2 and 4.3 with Proposition F.1 in Bücher and Kojadinovic (2013), we obtain that a test based on S_n whose p-value is computed using one of the two approaches in (4.6) will hold its level asymptotically as $n \to \infty$ followed by $M \to \infty$.

5 Simulation study

Large-scale Monte Carlo experiments were carried out in order to study the finite-sample performance of the derived tests for detecting changes in cross-sectional dependence. The main questions addressed by the study are the following:

(i) How well do the tests hold their size under the null hypothesis H_0 in (1.1) of no change?

(ii) What is the power of the tests against the alternative $H_{1,c}$ of a single change in cross-sectional dependence at constant margins? Specifically, the alternative hypothesis is $H_{1,c} \cap H_{0,m}$ with $H_{0,m}$ in (1.2) and $H_{1,c}$ defined by

$$\begin{aligned} H_{1,c} : \exists \text{ distinct } C_1 \text{ and } C_2, \text{ and } k^\star \in \{1, \dots, n-1\} \text{ such that} \\ \boldsymbol{X}_1, \dots, \boldsymbol{X}_{k^\star} \text{ have copula } C_1 \text{ and } \boldsymbol{X}_{k^\star+1}, \dots, \boldsymbol{X}_n \text{ have copula } C_2. \end{aligned} \tag{5.1}$$

(iii) What happens if the change in distribution is only due to a change in the margins, the copula remaining constant? Specifically, the alternative hypothesis is $H_{1,m} \cap H_{0,c}$ with $H_{0,c}$ given in (1.3) and $H_{1,m}$ defined by

$$\begin{aligned} H_{1,m} : \exists \text{ distinct } & F_{1,1}, F_{1,2} \text{ as well as } F_2, \dots, F_d \text{ and } k_1^\star \in \{1, \dots, n-1\} \\ & \text{such that } \boldsymbol{X}_1, \dots, \boldsymbol{X}_{k_1^\star} \text{ have marginal c.d.f.s } F_{1,1}, F_2, \dots, F_d \\ & \text{and } \boldsymbol{X}_{k_1^\star+1}, \dots, \boldsymbol{X}_n \text{ have marginal c.d.f.s } F_{1,2}, F_2, \dots, F_d. \end{aligned} \tag{5.2}$$

In addition to the three questions above, many others can be formulated, involving other alternative hypotheses for instance. The problem is complex and there are countless ways of combining factors in the experimental design. In our study, the settings were chosen to represent a wide and hopefully representative variety of situations, in function of the three questions above. The main factors of our experiments are summarized below:

- Test statistics:
  - Our statistic S_n in (2.4) with p-values computed via resampling using $\hat{S}_n^{(m)}$ or $\check{S}_n^{(m)}$ in (4.4) and (4.5), respectively. As we continue, we shall simply talk about the test based on $\hat{S}_n$ or $\check{S}_n$, respectively, to distinguish between these two situations.
  - The statistic S_n^R in (2.8) of Bücher and Ruppert (2013), with p-values computed according to the resampling method for $\mathbb{D}_n^R$ in (4.7).
- Sample size: $n \in \{50, 100, 200\}$.
- Number of samples per setting: 1 000.
- Cross-sectional dimension: $d \in \{2, 3\}$.
- Significance level: $\alpha = 5\%$.
- Serial dependence: The data were generated either as being serially independent or via two time-series models, an autoregressive process and a multivariate version of the exponential autoregressive model considered in Auestad and Tjøstheim (1990) and Paparoditis and Politis (2001, Section 3.3). Independent standard normals were used as multipliers for independent observations, while for the serially dependent datasets, the dependent multiplier sequences were generated from initial independent standard normal sequences using the "moving average approach" proposed initially in Bühlmann (1993) and revisited in some detail in Bücher and Kojadinovic (2013, Section 6.1). The value of the bandwidth parameter ℓ_n defined in Condition (M2) was chosen automatically using the approach described in Bücher and Kojadinovic (2013, Section 5). See Appendix D for details.
- Margins: in all but one setting, the margins were kept constant, i.e., $H_{0,m}$ in (1.2) was assumed. In one case (see Table 5), a break as in $H_{1,m}$ in (5.2) was assumed, the marginal distribution of the first component changing from the $N(0,1)$ to the $N(\mu, 1)$ distribution.
- Copulas: Clayton, Gumbel–Hougaard, Normal, Frank, with positive or negative (insofar possible) association, as well as asymmetric versions obtained via Khoudraji's device (Khoudraji, 1995; Genest et al., 1998; Liebscher, 2008).
- Alternative hypotheses involving a single change-point occurring at time $k^\star = \lfloor nt \rfloor$ with $t \in \{0.1, 0.25, 0.5, 0.75\}$:
  - $H_{0,m} \cap H_{1,c}$ with a change of the parameter within a copula family.
  - $H_{0,m} \cap H_{1,c}$ with a change of the copula family at constant Kendall's tau.

- $H_{1,m} \cap H_{0,c}$, i.e., a change of one of the *margins* rather than of the copula.
- For the serially dependent case, a change in the copula of the innovations, leading to a gradual change of the copula of the marginal distributions of the observables.

The experiments were carried out in the R statistical system (R Development Core Team, 2013) using the `copula` package (Hofert et al., 2013). To allow us to reuse previously written code, the rescaled ranks in (2.2) were computed by dividing the ranks by $l-k+2$ instead of $l-k+1$. Because (4.4) only involves rescaled ranks computed from the entire sample, the test based on $\hat{S}_n$ can be implemented to be substantially faster than the one based on $\check{S}_n$ for larger sample sizes. The corresponding routines are available in the R package `npcp` (Kojadinovic, 2014).

For the sake of brevity, only a representative subset of the results is reported here. Specifically, the following tables are provided in Appendix D:

- Size of the tests under the null hypothesis H_0:
  - Table 1: Percentage of false rejections when data are serially independent.
  - Table 2: Percentage of false rejections when data are serially dependent.
- Power of the tests against specific alternatives:
  - Table 3: Power against $H_{0,m} \cap H_{1,c}$ involving a change of the copula parameter within a copula family and at serial independence.
  - Table 4: Power against $H_{0,m} \cap H_{1,c}$ involving a change of copula family at a constant value of Kendall's tau and at serial independence.
  - Table 5: Power against $H_{1,m} \cap H_{0,c}$ involving a change in one of the margins and at serial independence..
  - Table 6: Power against $\neg H_0$ when data are serially dependent and the change occurs in the copula of the innovations.

Besides findings of a more anecdotical nature, the following conclusions may be drawn from the results:

- All tests hold their level reasonably well in the case of serial independence (Table 1), with minor fluctuations depending on sample size, test statistic, copula parameter and copula family.
- In case of serial dependence, the test based on $\hat{S}_n$ is too conservative for the sample sizes under consideration (Table 2). In line with this observation, the test based on $\check{S}_n$ appears to be more powerful than the one based on $\hat{S}_n$ (Table 6).
- For alternative hypotheses involving a change in the *copula*, the tests based on $\hat{S}_n$ and $\check{S}_n$ have a higher power than S_n^R (Tables 3 and 4). When the copula changes in such a way that Kendall's tau remains constant, the power of S_n^R is especially low. With respect to that last setting, note that distinguishing copulas on the basis of low amounts of data is known to be difficult (Genest et al., 2009; Kojadinovic et al., 2011b). The fact that the change-point is unknown makes the problem even harder.

- For alternative hypotheses involving a change in one of the *margins*, it is the test statistic S_n^R that is substantially more powerful than S_n (Table 5). The weak power of S_n can be explained by the fact that it is designed for detecting changes in the copula. Another tentative reading of the results is that the test based on S_n, regarded as a procedure for testing $H_{0,c}$, is relatively robust against small changes in one margin. In contrast, the test based on S_n^R behaves as an all-purpose test for the hypothesis of a constant distribution rather than as a test for a constant copula.

6 Case studies

As an illustration, we first applied the test based on $\check{S}_n$ to bivariate financial data consisting of daily logreturns computed from the DAX and the Standard and Poor 500 indices. Following Dehling et al. (2013, Section 7), attention was restricted to the years 2006–2009. The corresponding closing quotes were obtained from `http://quote.yahoo.com` using the `get.hist.quote` function of the `tseries` R package (Trapletti and Hornik, 2013), which resulted in $n = 993$ bivariate logreturns. Dependent multiplier sequences were generated as explained in Appendix C. An approximate p-value of 0.04 was obtained, providing some evidence against H_0. The conclusion is in line with the results reported in Dehling et al. (2013). Of course, as discussed earlier, it is only under the assumption that $H_{0,m}$ in (1.2) holds that it would be fully justified to decide to reject $H_{0,c}$ in (5.1) on the basis of the previous approximate p-value. The value of the change-point estimator $k_n^\star$ in (2.5) is 529, corresponding to February 22nd, 2008.

As a second illustration, we followed again Dehling et al. (2013) and considered $n = 504$ bivariate logreturns computed from closing daily quotes of the Dow Jones Industrial Average and the Nasdaq Composite for the years 1987 and 1988. The former quotes, not being available on `http://quote.yahoo.com` anymore, were taken from the R package `QRM` (Pfaff and McNeil, 2013). This two-year period is of interest because it contains October 19th, 1987, known as "black Monday" (see Dehling et al., 2013, Figure 4). An approximate p-value of 0.59 was obtained. Hence, despite the extreme events that occurred during the period under consideration, the test based on $\check{S}_n$ detects no evidence against H_0 in the data, which is in line with the results reported in Dehling et al. (2013).

7 Conclusion

We have demonstrated that the sensitivity of rank-based tests for the null hypothesis of a constant distribution against changes in cross-sectional dependence can be improved if ranks are computed with respect to relevant subsamples. In this way, the test we propose achieves in many cases a higher power than the one proposed in Bücher and Ruppert (2013). The limit distribution of the test statistic under the null hypothesis is unwieldy, but approximate p-values can still be computed via a multiplier resampling scheme. To deal with potential serial dependence, we make use of dependent multiplier sequences, an idea going back to Bühlmann (1993) and revisited in Bücher and Kojadinovic (2013).

Here are some potential avenues for further research:

- Once the null hypothesis has been rejected, the nature of the nonstationary needs to be investigated further: is there a single change-point or is there more than one? Or maybe the change is gradual rather than sudden? And does the change concern the margins or the copula?
- Can one detect a change in the copula without the hypothesis that the margins are constant?
- The procedure is computationally intensive because the ranks have to be recomputed for every $k \in \{1, \ldots, n-1\}$. Efficient algorithms for reutilizing calculations from one value of k to the next one might speed up the computations.

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References

B. Auestad and D. Tjøstheim. Identification of nonlinear time series: First order characterization and order determination. *Biometrika*, 77:669–687, 1990.

J. Bai. Weak convergence of the sequential empirical processes of residuals in ARMA models. *The Annals of Statistics*, 22(4):2051–2061, 1994.

A. Bücher. A note on nonparametric estimation of bivariate tail dependence. *Statistics and Risk Modeling*, page in press, 2013a. URL `http://www.ruhr-uni-bochum.de/imperia/md/content/mathematik3/publications/taildep_buecher.pdf`.

A. Bücher. A note on weak convergence of the sequential multivariate empirical process under strong mixing. *Journal of Theoretical Probability*, page in press, 2013b.

A. Bücher and H. Dette. A note on bootstrap approximations for the empirical copula process. *Statistics and Probability Letters*, 80(23–24):1925–1932, 2010.

A. Bücher and I. Kojadinovic. A dependent multiplier bootstrap for the sequential empirical copula process under strong mixing. *arXiv:1306.3930*, 2013.

A. Bücher and M. Ruppert. Consistent testing for a constant copula under strong mixing based on the tapered block multiplier technique. *Journal of Multivariate Analysis*, 116: 208–229, 2013.

A. Bücher and J. Segers. Extreme value copula estimation based on block maxima of a multivariate stationary time series. *arXiv:1311.3060*, 2013.

A. Bücher and S. Volgushev. Empirical and sequential empirical copula processes under serial dependence. *Journal of Multivariate Analysis*, 119:61–70, 2013.

P. Bühlmann. *The blockwise bootstrap in time series and empirical processes.* PhD thesis, ETH Zürich, 1993. Diss. ETH No. 10354.

M. Csörgő and L. Horváth. *Limit theorems in change-point analysis.* Wiley Series in Probability and Statistics. John Wiley & Sons, Chichester, UK, 1997.

H. Dehling, D. Vogel, M. Wendler, and D. Wied. An efficient and robust test for a change-point in correlation. *arXiv:1203.4871*, 2013.

C. Genest and J. Segers. On the covariance of the asymptotic empirical copula process. *Journal of Multivariate Analysis*, 101:1837–1845, 2010.

C. Genest, K. Ghoudi, and L.-P. Rivest. Discussion of "Understanding relationships using copulas", by E. Frees and E. Valdez. *North American Actuarial Journal*, 3:143–149, 1998.

C. Genest, B. Rémillard, and D. Beaudoin. Goodness-of-fit tests for copulas: A review and a power study. *Insurance: Mathematics and Economics*, 44:199–213, 2009.

E. Gombay and L. Horváth. Change-points and bootstrap. *Environmetrics*, 10(6), 1999.

E. Gombay and L. Horváth. Rates of convergence for U-statistic processes and their bootstrapped versions. *Journal of Statistical Planning and Inference*, 102:247–272, 2002.

M. Hofert, I. Kojadinovic, M. Mächler, and J. Yan. *copula: Multivariate dependence with copulas*, 2013. URL `http://CRAN.R-project.org/package=copula`. R package version 0.999-7.

M. Holmes, I. Kojadinovic, and J-F. Quessy. Nonparametric tests for change-point detection à la Gombay and Horváth. *Journal of Multivariate Analysis*, 115:16–32, 2013.

A. Inoue. Testing for distributional change in time series. *Econometric Theory*, 17(1): 156–187, 2001.

A. Khoudraji. *Contributions à l'étude des copules et à la modélisation des valeurs extrêmes bivariées.* PhD thesis, Université Laval, Québec, Canada, 1995.

I. Kojadinovic. *npcp: Some nonparametric tests for change-point detection in (multivariate) observations*, 2014. URL `http://CRAN.R-project.org/package=npcp`. R package version 0.0-1.

I. Kojadinovic, J. Segers, and J. Yan. Large-sample tests of extreme-value dependence for multivariate copulas. *The Canadian Journal of Statistics*, 39(4):703–720, 2011a.

I. Kojadinovic, J. Yan, and M. Holmes. Fast large-sample goodness-of-fit for copulas. *Statistica Sinica*, 21(2):841–871, 2011b.

E. Liebscher. Construction of asymmetric multivariate copulas. *Journal of Multivariate Analysis*, 99:2234–2250, 2008.

E. Paparoditis and D.N. Politis. Tapered block bootstrap. *Biometrika*, 88(4):1105–1119, 2001.

B. Pfaff and A. McNeil. *QRM: Provides R-language Code to Examine Quantitative Risk Management Concepts*, 2013. URL `http://CRAN.R-project.org/package=QRM`. R package version 0.4-9.

J.-F. Quessy, M. Saïd, and A.-C. Favre. Multivariate Kendall's tau for change-point detection in copulas. *The Canadian Journal of Statistics*, 41:65–82, 2013.

R Development Core Team. *R: A Language and Environment for Statistical Computing.* R Foundation for Statistical Computing, Vienna, Austria, 2013. URL `http://www.R-project.org`. ISBN 3-900051-07-0.

B. Rémillard. Goodness-of-fit tests for copulas of multivariate time series. *Social Science Research Network*, 1729982:1–32, 2010. URL `http://ssrn.com/abstract=1729982`.

B. Rémillard and O. Scaillet. Testing for equality between two copulas. *Journal of Multivariate Analysis*, 100(3):377–386, 2009.

J.P. Romano and M. Wolf. A more general central limit theorem for m-dependent random variables with unbounded m. *Statistics and Probability Letters*, 47:115–124, 2000.

O. Scaillet. A Kolmogorov-Smirnov type test for positive quadrant dependence. *Canadian Journal of Statistics*, 33:415–427, 2005.

J. Segers. Asymptotics of empirical copula processes under nonrestrictive smoothness assumptions. *Bernoulli*, 18:764–782, 2012.

A. Sklar. Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Institut de Statistique de l'Université de Paris*, 8:229–231, 1959.

A. Trapletti and K. Hornik. *tseries: Time series analysis and computational finance*, 2013. URL `http://CRAN.R-project.org/package=tseries`. R package version 0.10-32.

A.W. van der Vaart and J.A. Wellner. *Weak convergence and empirical processes.* Springer, New York, 2000. Second edition.

M. van Kampen and D. Wied. A nonparametric constancy test for copulas under mixing conditions. *working paper*, 2013. URL `http://www.statistik.tu-dortmund.de/vankampen-eng.html`.

D. Wied, H. Dehling, M. van Kampen, and D. Vogel. A fluctuation test for constant Spearman's rho with nuisance-free limit distribution. *Computational Statistics and Data Analysis*, page in press, 2013.

A Proof of Proposition 3.3

Let us first introduce additional notation. For integers $1 \leq k \leq l \leq n$, let $H_{k:l}$ denote the empirical c.d.f. of the unobservable sample $\boldsymbol{U}_k, \dots, \boldsymbol{U}_l$ and let $H_{k:l,j}$, for $j \in \{1, \dots, d\}$, denote its margins. The empirical quantile functions are

$$H_{k:l,j}^{-1}(u) = \inf\{v \in [0,1] : H_{k:l,j}(v) \geq u\}, \qquad u \in [0,1],$$

which are collected in a vector via

$$\boldsymbol{H}_{k:l}^{-1}(\boldsymbol{u}) = \big(H_{k:l,1}^{-1}(u_1), \dots, H_{k:l,d}^{-1}(u_d)\big), \qquad \boldsymbol{u} \in [0,1]^d.$$

By convention, the previously defined quantities are all taken equal to zero if $k > l$.

From the proof of Theorem 3.4 in Bücher and Kojadinovic (2013), we have that (3.6) holds with $\mathbb{C}_n$ replaced by $\mathbb{C}_n^{\mathrm{alt}}$, where

$$\mathbb{C}_n^{\mathrm{alt}}(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} [\mathbf{1}\{\boldsymbol{U}_i \leq \boldsymbol{H}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}^{-1}(\boldsymbol{u})\} - C(\boldsymbol{u})], \qquad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d.$$

To show (3.6), it remains therefore to prove that

$$\sup_{(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d} \left| \mathbb{C}_n^{\mathrm{alt}}(s,t,\boldsymbol{u}) - \mathbb{C}_n(s,t,\boldsymbol{u}) \right| \xrightarrow{\mathrm{P}} 0. \tag{A.1}$$

To do so, we adapt the arguments used in Lemma A.2 of Bücher and Segers (2013). Fix $1 \leq k \leq l \leq n$ and $\boldsymbol{u} \in [0,1]^d$. For $i \in \{k, \dots, l\}$, the d components of $\hat{\boldsymbol{U}}_i^{k:l}$ defined in (2.2) can be expressed as $\hat{U}_{ij}^{k:l} = H_{k:l,j}(U_{ij})$, $j \in \{1, \dots, d\}$. Next, notice that

$$\begin{aligned} \mathbf{1}\big\{U_{ij} \leq H_{k:l,j}^{-1}(u_j)\big\} - \mathbf{1}\big(\hat{U}_{ij}^{k:l} \leq u_j\big) &= \mathbf{1}\big\{U_{ij} < H_{k:l,j}^{-1}(u_j)\big\} + \mathbf{1}\big\{U_{ij} = H_{k:l,j}^{-1}(u_j)\big\} \\ &\quad - \mathbf{1}\big(\hat{U}_{ij}^{k:l} < u_j\big) - \mathbf{1}\big(\hat{U}_{ij}^{k:l} = u_j\big) \\ &= \mathbf{1}\big\{U_{ij} = H_{k:l,j}^{-1}(u_j)\big\} - \mathbf{1}\big(\hat{U}_{ij}^{k:l} = u_j\big), \end{aligned}$$

as $x < H^{-1}(u)$ if and only $H(x) < u$ for any distribution function H. Since $\hat{U}_{ij}^{k:l} = H_{k:l,j}(U_{ij}) = u_j$ implies $U_{ij} = H_{k:l,j}^{-1}(u_j)$, we obtain that

$$0 \leq \mathbf{1}\big\{U_{ij} \leq H_{k:l,j}^{-1}(u_j)\big\} - \mathbf{1}\big(\hat{U}_{ij}^{k:l} \leq u_j\big) \leq \mathbf{1}\big\{U_{ij} = H_{k:l,j}^{-1}(u_j)\big\}.$$

Combining the previous inequality with the decomposition

$$\begin{aligned} &\mathbf{1}\{\boldsymbol{U}_i \leq \boldsymbol{H}_{k:l}^{-1}(\boldsymbol{u})\} - \mathbf{1}(\hat{\boldsymbol{U}}_i^{k:l} \leq \boldsymbol{u}) \\ &\qquad = \sum_{p=1}^{d} \Bigg[\prod_{1 \leq j \leq p} \mathbf{1}\{U_{ij} \leq H_{k:l,j}^{-1}(u_j)\} \prod_{p < j \leq d} \mathbf{1}(\hat{U}_{ij}^{k:l} \leq u_j) \\ &\qquad\qquad - \prod_{1 \leq j \leq p-1} \mathbf{1}\{U_{ij} \leq H_{k:l,j}^{-1}(u_j)\} \prod_{p-1 < j \leq d} \mathbf{1}(\hat{U}_{ij}^{k:l} \leq u_j) \Bigg], \end{aligned}$$

we obtain that

$$0 \le \mathbf{1}\{\boldsymbol{U}_i \le \boldsymbol{H}_{k:l}^{-1}(\boldsymbol{u})\} - \mathbf{1}(\hat{\boldsymbol{U}}_i^{k:l} \le \boldsymbol{u}) \le \sum_{j=1}^{d} \mathbf{1}\big\{U_{ij} = H_{k:l,j}^{-1}(u_j)\big\}. \tag{A.2}$$

It follows that the supremum in (A.1) is smaller than

$$\sum_{j=1}^{d} \sup_{(s,t,u)\in\Delta\times[0,1]} \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \mathbf{1}\big\{U_{ij} = H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}^{-1}(u)\big\} \le \sum_{j=1}^{d} \sup_{u\in[0,1]} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{1}(U_{ij} = u).$$

Using the fact that $\mathbf{1}\big(U_{ij} = u\big) \le \mathbf{1}\big(U_{ij} \le u\big) - \mathbf{1}\big(U_{ij} \le u - 1/n\big)$, the latter is smaller

$$d \sup_{\substack{\boldsymbol{u},\boldsymbol{v}\in[0,1]^d \\ \|\boldsymbol{u}-\boldsymbol{v}\|_1 \le n^{-1}}} |\mathbb{B}_n(0,1,\boldsymbol{u}) - \mathbb{B}_n(0,1,\boldsymbol{v})| + dn^{-1/2},$$

where $\mathbb{B}_n$ is defined in (3.4). Using the asymptotic uniform equicontinuity in probability of $\mathbb{B}_n$ established in Lemma 2 of Bücher (2013b), we finally obtain (A.1), which completes the proof. ■

B Proof of Proposition 4.3

We shall only prove the result in the case of strongly mixing observations, that is, when Condition 4.1(ii) is assumed. The proof is similar but simpler when Condition 4.1(i) is assumed instead.

It is sufficient to show the statement involving $\check{\mathbb{C}}_n^{(m)}$. The statements for $\check{\mathbb{D}}_n^{(m)}$ and $\check{S}_n^{(m)}$ then follow from the continuous mapping theorem.

For any $m \in \{1,\dots,M\}$ and $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$, put

$$\mathbb{B}_n^{(m)}(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi_{i,n}^{(m)} \{\mathbf{1}(\boldsymbol{U}_i \le \boldsymbol{u}) - C(\boldsymbol{u})\}, \tag{B.1}$$

$$\mathbb{C}_n^{(m)}(s,t,\boldsymbol{u}) = \mathbb{B}_n^{(m)}(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u})\, \mathbb{B}_n^{(m)}(s,t,\boldsymbol{u}^{(j)}).$$

[Recall that $\boldsymbol{u}^{(j)} = (1,\dots,1,u_j,1,\dots,1) \in [0,1]^d$, with u_j appearing at the j-th coordinate.] From Theorem 2.1 in Bücher and Kojadinovic (2013), we have that

$$\big(\mathbb{B}_n, \mathbb{B}_n^{(1)}, \dots, \mathbb{B}_n^{(M)}\big) \rightsquigarrow \big(\mathbb{B}_C, \mathbb{B}_C^{(1)}, \dots, \mathbb{B}_C^{(M)}\big)$$

in $\{\ell^\infty(\Delta \times [0,1]^d)\}^{M+1}$, where $\mathbb{B}_C^{(1)}, \dots, \mathbb{B}_C^{(M)}$ are independent copies of $\mathbb{B}_C$ in (3.5), and thus, from the continuous mapping theorem and (3.6), we find that

$$\big(\mathbb{C}_n, \mathbb{C}_n^{(1)}, \dots, \mathbb{C}_n^{(M)}\big) \rightsquigarrow \big(\mathbb{C}_C, \mathbb{C}_C^{(1)}, \dots, \mathbb{C}_C^{(M)}\big)$$

in $\{\ell^\infty(\Delta \times [0,1]^d)\}^{M+1}$. It is therefore sufficient to show that

$$\sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \big|(\check{\mathbb{C}}_n^{(m)} - \mathbb{C}_n^{(m)})(s,t,\boldsymbol{u})\big| \xrightarrow{\mathrm{P}} 0 \tag{B.2}$$

for every $m \in \{1,\dots,M\}$. Below, we will show the following two assertions: first,

$$\sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \big|(\check{\mathbb{B}}_n^{(m)} - \mathbb{B}_n^{(m)})(s,t,\boldsymbol{u})\big| \xrightarrow{\mathrm{P}} 0, \tag{B.3}$$

and second, for every $\delta \in (0,1/2)$ and every $\varepsilon \in (0,1)$,

$$\sup_{\substack{\boldsymbol{u}\in[0,1]^d\\ \delta\le u_j\le 1-\delta}} \sup_{\substack{(s,t)\in[0,1]^2\\ t-s\ge\varepsilon}} \big|\dot{C}_{j,\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - \dot{C}_j(\boldsymbol{u})\big| \xrightarrow{\mathrm{P}} 0. \tag{B.4}$$

In view of the structure of $\check{\mathbb{C}}_n^{(m)}$ in (4.3), the assertions (B.3) and (B.4) imply (B.2), as we show next. Clearly,

$$\begin{aligned}
&\big|(\check{\mathbb{C}}_n^{(m)} - \mathbb{C}_n^{(m)})(s,t,\boldsymbol{u})\big| \\
&\quad\le \big|(\check{\mathbb{B}}_n^{(m)} - \mathbb{B}_n^{(m)})(s,t,\boldsymbol{u})\big| + \sum_{j=1}^d \big|\dot{C}_{j,\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u})\big| \, \big|(\check{\mathbb{B}}_n^{(m)} - \mathbb{B}_n^{(m)})(s,t,\boldsymbol{u}^{(j)})\big| \\
&\qquad + \sum_{j=1}^d \big|\dot{C}_{j,\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - \dot{C}_j(\boldsymbol{u})\big| \, \big|\mathbb{B}_n^{(m)}(s,t,\boldsymbol{u}^{(j)})\big|.
\end{aligned} \tag{B.5}$$

Taking suprema over $(s,t,\boldsymbol{u}) \in \Delta\times[0,1]^d$, the first and the second term on the right-hand side of (B.5) converge to zero in probability because of assertion (B.3) and uniform boundedness of $\dot{C}_{j,k:l}$ (see Kojadinovic et al., 2011a, proof of Proposition 2). The third term on the right-hand side of (B.5) converges to zero in probability because of assertion (B.4) and the fact that $(s,t,\boldsymbol{u}) \mapsto \mathbb{B}_n^{(m)}(s,t,\boldsymbol{u}^{(j)})$ vanishes as soon as $s = t$ or $u_j \in \{0,1\}$, and is asymptotically uniformly equicontinuous in probability as a consequence of Lemma A.3 in Bücher and Kojadinovic (2013).

It remains to show (B.3) and (B.4). The proof of the latter assertion is simplest and is given first.

Proof of (B.4). Observe that

$$C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) = C(\boldsymbol{u}) + \frac{1}{\sqrt{n}\,\lambda_n(s,t)}\,\mathbb{C}_n(s,t,\boldsymbol{u}).$$

Fix $\delta \in (0,1/2)$ and $\varepsilon \in (0,1)$. Without loss of generality, assume that n is large enough so that the bandwidth $h = h_n(s,t) = 1/\sqrt{\lfloor nt\rfloor - \lfloor ns\rfloor}$ is less than δ whenever $t - s \ge \varepsilon$. Then, for $t-s\ge\varepsilon$ and $\boldsymbol{u} \in [0,1]^d$ with $\delta \le u_j \le 1-\delta$, we have

$$\begin{aligned}
\dot{C}_{j,\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) &= \frac{1}{2h}\{C(\boldsymbol{u}+h\boldsymbol{e}_j) - C(\boldsymbol{u}-h\boldsymbol{e}_j)\} \\
&\qquad + \frac{1}{2h\,\sqrt{n}\,\lambda_n(s,t)}\{\mathbb{C}_n(s,t,\boldsymbol{u}+h\boldsymbol{e}_j) - \mathbb{C}_n(s,t,\boldsymbol{u}-h\boldsymbol{e}_j)\}.
\end{aligned}$$

By the assumption of existence and continuity of $\dot{C}_j$ on V_j (see Condition 3.2), and since $0 \le \dot{C}_j \le 1$, it follows from the mean-value theorem that

$$\sup_{\substack{\boldsymbol{u}\in[0,1]^d \\ \delta\le u_j\le 1-\delta}} \left| \frac{1}{2h}\{C(\boldsymbol{u}+h\boldsymbol{e}_j) - C(\boldsymbol{u}-h\boldsymbol{e}_j)\} - \dot{C}_j(\boldsymbol{u}) \right| \to 0, \qquad h \to 0.$$

Using (3.6) and the fact that $\mathbb{B}_n$ is asymptotically uniformly equicontinuous in probability, it can be verified that $\mathbb{C}_n$ is asymptotically uniformly equicontinuous in probability as well. It follows that

$$\sup_{\substack{\boldsymbol{u}\in[0,1]^d \\ \delta\le u_j\le 1-\delta}} \sup_{\substack{(s,t)\in[0,1]^2 \\ t-s\ge\varepsilon}} \left|\mathbb{C}_n(s,t,\boldsymbol{u}+h\boldsymbol{e}_j) - \mathbb{C}_n(s,t,\boldsymbol{u}-h\boldsymbol{e}_j)\right| \xrightarrow{\mathrm{P}} 0.$$

Finally,

$$\frac{1}{2h\sqrt{n}\,\lambda_n(s,t)} = \frac{1}{2\sqrt{\lambda_n(s,t)}} \le \frac{1}{2\sqrt{\varepsilon - 1/n}}.$$

Combine the four previous displays to arrive at the desired conclusion. ■

The proof of (B.3) is more complicated. Using the notation introduced in Appendix A, let us define the auto-centered version of the process $\mathbb{B}_n^{(m)}$ in (B.1) as

$$\begin{aligned}\mathring{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) &= \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi_{i,n}^{(m)} \{\mathbf{1}(\boldsymbol{U}_i \le \boldsymbol{u}) - H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u})\} \\ &= \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (\xi_{i,n}^{(m)} - \bar{\xi}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{(m)})\, \mathbf{1}(\boldsymbol{U}_i \le \boldsymbol{u}),\end{aligned} \tag{B.6}$$

with the usual convention that empty sums are zero.

Proof of (B.3). Consider the decomposition

$$\begin{aligned}\left|(\check{\mathbb{B}}_n^{(m)} - \mathbb{B}_n^{(m)})(s,t,\boldsymbol{u})\right| \le{}& \left|\check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) - \mathring{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{H}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{-1}(\boldsymbol{u}))\right| \\ &+ \left|\mathring{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{H}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{-1}(\boldsymbol{u})) - \mathbb{B}_n^{(m)}(s,t,\boldsymbol{H}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{-1}(\boldsymbol{u}))\right| \\ &+ \left|\mathbb{B}_n^{(m)}(s,t,\boldsymbol{H}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{-1}(\boldsymbol{u})) - \mathbb{B}_n^{(m)}(s,t,\boldsymbol{u})\right|.\end{aligned}$$

Write out the definitions of the processes $\mathbb{B}_n^{(m)}$, $\mathring{\mathbb{B}}_n^{(m)}$ and $\check{\mathbb{B}}_n^{(m)}$ in (B.1), (B.6) and (4.2), respectively, and take suprema over $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$ to obtain

$$\sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left|(\check{\mathbb{B}}_n^{(m)} - \mathbb{B}_n^{(m)})(s,t,\boldsymbol{u})\right|$$

$$\le \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left|\check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) - \mathring{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{H}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{-1}(\boldsymbol{u}))\right| \tag{B.7}$$

$$+ \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left|\frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi_{i,n}^{(m)}\right| \left|H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - C(\boldsymbol{u})\right| \tag{B.8}$$

$$+ \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left|\mathbb{B}_n^{(m)}(s,t,\boldsymbol{H}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{-1}(\boldsymbol{u})) - \mathbb{B}_n^{(m)}(s,t,\boldsymbol{u})\right|. \tag{B.9}$$

Each term requires a different treatment.

1- *The term* (B.8)*:* Let $\delta \in (0,1/2)$, to be specified later. We split the supremum into two parts, according to whether $t-s$ is smaller or larger than $a_n = n^{-1/2-\delta}$:

$$A_{n,1} = \sup_{\substack{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d \\ t-s\le a_n}} \left| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi_{i,n}^{(m)} \right| \left| H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - C(\boldsymbol{u}) \right|,$$

$$A_{n,2} = \sup_{\substack{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d \\ t-s\ge a_n}} \left| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi_{i,n}^{(m)} \right| \left| H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - C(\boldsymbol{u}) \right|.$$

We will show that both $A_{n,1}$ and $A_{n,2}$ converge to zero in probability.

(a) Since both $H_{k:l}$ and C take values in $[0,1]$, a crude bound for $A_{n,1}$ is

$$A_{n,1} \le \frac{1}{\sqrt{n}}(n\,a_n + 1) \max_{1\le i\le n} |\xi_{i,n}^{(m)}| \le 2\,n^{-\delta} \max_{1\le i\le n} |\xi_{i,n}|.$$

Using the fact that, from (M1), for any $\nu \ge 1$, $\sup_{n\ge 1} \mathrm{E}[|\xi_{1,n}^{(m)}|^\nu] < \infty$, we have that, for every $\alpha > 0$ and $\nu \ge 1$ such that $\nu > 1/\alpha$,

$$\mathrm{P}\left(\max_{1\le i\le n} |\xi_{i,n}^{(m)}| \ge n^\alpha \right) \le n\mathrm{P}\big(|\xi_{1,n}^{(m)}| \ge n^\alpha \big) \le n^{1-\nu\alpha}\ \mathrm{E}[|\xi_{1,n}^{(m)}|^\nu] \to 0.$$

Apply the previous display with $\alpha \in (0,\delta)$ to find that $A_{n,1}$ converges to zero in probability.

(b) Recall $\mathbb{B}_n$ in (3.4). Observe that

$$\mathbb{B}_n(s,t,\boldsymbol{u}) = \frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{\sqrt{n}} \{H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - C(\boldsymbol{u})\}.$$

We have

$$\begin{aligned} A_{n,2} &= \sup_{\substack{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d \\ t-s\ge a_n}} \left| \frac{1}{\lfloor nt\rfloor - \lfloor ns\rfloor} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi_{i,n}^{(m)} \right| \left| \mathbb{B}_n(s,t,\boldsymbol{u}) \right| \\ &\le \sup_{\substack{\lfloor ns\rfloor<\lfloor nt\rfloor \\ \lfloor nt\rfloor+1-\lfloor ns\rfloor\ge na_n}} \left| \frac{1}{\lfloor nt\rfloor - \lfloor ns\rfloor} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi_{i,n}^{(m)} \right| \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left| \mathbb{B}_n(s,t,\boldsymbol{u}) \right| \\ &\le \max_{\substack{1\le k\le l\le n \\ l-k\ge na_n}} \left| \frac{1}{l-k+1} \sum_{i=k}^{l} \xi_{i,n}^{(m)} \right| \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left| \mathbb{B}_n(s,t,\boldsymbol{u}) \right|. \end{aligned}$$

By weak convergence $\mathbb{B}_n \rightsquigarrow \mathbb{B}_C$ in $\ell^\infty(\Delta\times[0,1]^d)$, the supremum at the end of the previous display is bounded in probability. Writing $b_n = na_n$, it is sufficient to show that

$$\max_{\substack{1\le k\le l\le n \\ l-k\ge b_n}} \left| \frac{1}{l-k+1} \sum_{i=k}^{l} \xi_{i,n}^{(m)} \right| \xrightarrow{\mathrm{P}} 0, \qquad n\to\infty.$$

Fix $\eta > 0$. The probability that the previous maximum exceeds η is bounded by

$$\sum_{\substack{1\le k\le l\le n\\ l-k\ge b_n}} \mathrm{P}\left[\left|\frac{1}{l-k+1}\sum_{i=k}^{l}\xi_{i,n}^{(m)}\right| > \eta\right]. \tag{B.10}$$

Fix $\nu \ge 2$, to be specified later. By stationarity and Markov's inequality, the previous expression is bounded by

$$\sum_{\substack{1\le k\le l\le n\\ l-k\ge b_n}} \eta^{-\nu}\,(l-k+1)^{-\nu}\ \mathrm{E}\left[\left|\sum_{i=1}^{l-k+1}\xi_{i,n}^{(m)}\right|^{\nu}\right] \le \eta^{-\nu}\,n \sum_{b_n\le r\le n} r^{-\nu}\ \mathrm{E}\left[\left|\sum_{i=1}^{r}\xi_{i,n}^{(m)}\right|^{\nu}\right].$$

Recall that the sequence $(\xi_{i,n}^{(m)})_{i\in\mathbb{Z}}$ is ℓ_n-dependent from (M2) and assume that n is sufficiently large so that $n \ge 2\ell_n + b_n$. Then, by Corollary A.1 in Romano and Wolf (2000), there exists a constant C_ν, depending only on ν, such that

$$\mathrm{E}\left[\left|\sum_{i=1}^{r}\xi_{i,n}^{(m)}\right|^{\nu}\right] \le C_\nu^{\nu}\,(4\ell_n r)^{\nu/2}\ \mathrm{E}[|\xi_{1,n}^{(m)}|^{\nu}].$$

Using the fact that, from (M1), $\sup_{n\ge1}\mathrm{E}[|\xi_{1,n}^{(m)}|^{\nu}] < \infty$, and up to a multiplicative constant, the expression in (B.10) is bounded by

$$\begin{aligned} n\sum_{b_n\le r\le n} r^{-\nu}(\ell_n r)^{\nu/2} &\le n^2 b_n^{-\nu/2}\ell_n^{\nu/2} \\ &= O\big(n^{2-(1/2-\delta)\nu/2+(1/2-\gamma)\nu/2}\big) \\ &= O\big(n^{2+(\delta-\gamma)\nu/2}\big). \end{aligned}$$

The right-hand side converges to zero if we choose $\delta = \gamma/2$ and then $\nu > 8/\gamma$.

2- *The term* (B.9)*:* We have to show that, for every $\eta, \lambda > 0$,

$$\mathrm{P}\left[\sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d}\big|\mathbb{B}_n^{(m)}\{s,t,\boldsymbol{H}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{-1}(\boldsymbol{u})\} - \mathbb{B}_n^{(m)}(s,t,\boldsymbol{u})\big| > \lambda\right] \le \eta,$$

for all sufficiently large n.

Fix $\eta, \lambda > 0$. Since (B.9) is smaller than $2\sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d}|\mathbb{B}_n^{(m)}(s,t,\boldsymbol{u})|$, and using the fact that $\mathbb{B}_n^{(m)}$ vanishes on the diagonal $s = t$ and is asymptotically uniformly equicontinuous in probability, there exists $\varepsilon \in (0,1)$ such that, for all n sufficiently large,

$$\mathrm{P}\left[\sup_{\substack{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d\\ t-s<\varepsilon}}\big|\mathbb{B}_n^{(m)}\{s,t,\boldsymbol{H}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{-1}(\boldsymbol{u})\} - \mathbb{B}_n^{(m)}(s,t,\boldsymbol{u})\big| > \lambda\right] \le \eta/2.$$

Setting $\zeta_n = \sup_{\substack{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d \\ t-s\geq\varepsilon}} \|\boldsymbol{H}^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - \boldsymbol{u}\|_1$, we shall now show that, for all n sufficiently large,

$$A_n = \mathrm{P}\left\{ \sup_{\substack{(s,t,\boldsymbol{u},\boldsymbol{v})\in\Delta\times[0,1]^{2d} \\ t-s\geq\varepsilon,\, \|\boldsymbol{u}-\boldsymbol{v}\|_1\leq\zeta_n}} \left|\mathbb{B}_n^{(m)}(s,t,\boldsymbol{u}) - \mathbb{B}_n^{(m)}(s,t,\boldsymbol{v})\right| > \lambda \right\} \leq \eta/2,$$

which will complete the proof. Using again the asymptotic uniformly equicontinuity in probability of $\mathbb{B}_n^{(m)}$, there exists $\mu \in (0,1)$ such that, for all n sufficiently large,

$$A_{n,1} = \mathrm{P}\left\{ \sup_{\substack{(s,t,\boldsymbol{u},\boldsymbol{v})\in\Delta\times[0,1]^{2d} \\ t-s\geq\varepsilon,\, \|\boldsymbol{u}-\boldsymbol{v}\|_1\leq\mu}} \left|\mathbb{B}_n^{(m)}(s,t,\boldsymbol{u}) - \mathbb{B}_n^{(m)}(s,t,\boldsymbol{v})\right| > \lambda \right\} \leq \eta/4.$$

We then bound A_n by $A_{n,1}+A_{n,2}$, where $A_{n,2} = \mathrm{P}(\zeta_n > \mu)$. From the weak convergence of $\mathbb{B}_n$ to $\mathbb{B}_C$ in $\ell^\infty(\Delta\times[0,1]^d)$, we have that

$$\sup_{\substack{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d \\ t-s\geq\varepsilon}} |H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - C(\boldsymbol{u})| \\ \leq \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} |\mathbb{B}_n(s,t,\boldsymbol{u})| \times n^{-1/2} \times \sup_{\substack{(s,t)\in\Delta \\ t-s\geq\varepsilon}} \{\lambda_n(s,t)\}^{-1} \xrightarrow{\mathrm{P}} 0.$$

Using the fact that $\sup_{u\in[0,1]} |H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u) - u| = \sup_{u\in[0,1]} |H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u) - u|$ for $j\in\{1,\dots,d\}$ (for instance, by symmetry arguments on the graphs of $H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}$ and $H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}$), we immediately obtain that $\zeta_n \xrightarrow{\mathrm{P}} 0$, which implies that, for all n sufficiently large, $A_{n,2} \leq \eta/4$, and thus that, for all n sufficiently large, $A_n \leq \eta/2$.

3- *The term* (B.7)*:* For the following arguments, it is sufficient to assume that the sequence $(\xi_{i,n}^{(m)})_{i\in\mathbb{Z}}$ appearing in $\breve{\mathbb{B}}_n^{(m)}$ and $\mathring{\mathbb{B}}_n^{(m)}$ satisfies only (M1) with $\mathrm{E}[\{\xi_{0,n}^{(m)}\}^2] > 0$ not necessarily equal to one.

Let $K > 0$ be a constant and let us first suppose that, for any $n \geq 1$ and $i\in\{1,\dots,n\}$, $\xi_{i,n}^{(m)} \geq -K$. With (A.2) in mind, the term (B.7) is smaller than $A_{n,1}+A_{n,2}$, where

$$A_{n,1} = \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (\xi_{i,n}^{(m)}+K) \left[\mathbf{1}\{\boldsymbol{U}_i \leq \boldsymbol{H}^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u})\} - \mathbf{1}(\hat{\boldsymbol{U}}_i^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \leq \boldsymbol{u})\right]$$

and

$$A_{n,2} = \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \frac{K + \bar{\xi}^{(m)}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left[\mathbf{1}\{\boldsymbol{U}_i \leq \boldsymbol{H}^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u})\} - \mathbf{1}(\hat{\boldsymbol{U}}_i^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \leq \boldsymbol{u})\right].$$

Let us first show that $A_{n,1} \xrightarrow{\mathrm{P}} 0$. Plugging (A.2) into the expression of $A_{n,1}$, we bound $A_{n,1}$ by $A_{n,1,1}+\dots+A_{n,1,d}$, where

$$A_{n,1,j} = \sup_{(s,t,u)\in\Delta\times[0,1]} \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (\xi_{i,n}^{(m)}+K)\mathbf{1}(U_{ij} = u).$$

To prove that $A_{n,1} \xrightarrow{\mathrm{P}} 0$, we shall now show that $A_{n,1,j} \xrightarrow{\mathrm{P}} 0$ for all $j \in \{1,\dots,d\}$. Fix $j \in \{1,\dots,d\}$. Using the fact that $\mathbf{1}(U_{ij} = u) \leq \mathbf{1}(U_{ij} \leq u) - \mathbf{1}(U_{ij} \leq u - 1/n)$, we obtain that $A_{n,1,j}$ is smaller than $A'_{n,1,j} + A''_{n,1,j} + A'''_{n,1,j}$, where

$$A'_{n,1,j} = \sup_{(s,t,u)\in\Delta\times[0,1]} \left| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (\xi_{i,n}^{(m)} - \bar{\xi}^{(m)}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}) \left\{ \mathbf{1}\big(U_{ij} \leq u\big) - \mathbf{1}\big(U_{ij} \leq u - 1/n\big) \right\} \right|,$$

$$A''_{n,1,j} = K \sup_{(s,t,u)\in\Delta\times[0,1]} \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left\{ \mathbf{1}\big(U_{ij} \leq u\big) - \mathbf{1}\big(U_{ij} \leq u - 1/n\big) \right\},$$

$$A'''_{n,1,j} = \sup_{(s,t,u)\in\Delta\times[0,1]} \frac{\left|\bar{\xi}^{(m)}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}\right|}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left\{ \mathbf{1}\big(U_{ij} \leq u\big) - \mathbf{1}\big(U_{ij} \leq u - 1/n\big) \right\}.$$

From Lemma A.3 of Bücher and Kojadinovic (2013), we know that $\mathbb{B}_n^{(m)}$ is asymptotically uniformly equicontinuous in probability under the weaker conditions on the sequence $(\xi_{i,n}^{(m)})_{i\in\mathbb{Z}}$ considered above. The treatment of the term (B.8) carried out previously remains valid under these conditions and ensures that

$$\sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left| \mathring{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) - \mathbb{B}_n^{(m)}(s,t,\boldsymbol{u}) \right| \xrightarrow{\mathrm{P}} 0,$$

which implies that $\mathring{\mathbb{B}}_n^{(m)}$ is asymptotically uniformly equicontinuous in probability as well under the same weaker conditions on the sequence $(\xi_{i,n}^{(m)})_{i\in\mathbb{Z}}$. The latter immediately implies that $A'_{n,1,j} \xrightarrow{\mathrm{P}} 0$. For $A''_{n,1,j}$, we have

$$A''_{n,1,j} \leq K \sup_{\substack{\boldsymbol{u},\boldsymbol{v}\in[0,1]^d \\ \|\boldsymbol{u}-\boldsymbol{v}\|_1 \leq n^{-1}}} |\mathbb{B}_n(0,1,\boldsymbol{u}) - \mathbb{B}_n(0,1,\boldsymbol{v})| + Kn^{-1/2} \xrightarrow{\mathrm{P}} 0,$$

by asymptotic uniform equicontinuity in probability of $\mathbb{B}_n$. The fact that $A'''_{n,1,j} \xrightarrow{\mathrm{P}} 0$ can be shown by proceeding as for the term (B.8). Hence, we have that $A_{n,1,j} \xrightarrow{\mathrm{P}} 0$, which implies that $A_{n,1} \xrightarrow{\mathrm{P}} 0$.

The fact that $A_{n,2} \xrightarrow{\mathrm{P}} 0$, follows from (A.2) which implies that $A_{n,2}$ is smaller than $\sum_{j=1}^d (A''_{n,1,j} + A'''_{n,1,j})$. This completes the proof under the condition $\xi_{i,n}^{(m)} \geq -K$.

To show that this condition is not necessary, we proceed as at the end of the proof of Lemma A.3 of Bücher and Kojadinovic (2013). Let $Z_{i,n}^+ = \max(\xi_{i,n}^{(m)}, 0)$, $Z_{i,n}^- = \max(-\xi_{i,n}^{(m)}, 0)$, $K^+ = \mathrm{E}(Z_{0,n}^+)$ and $K^- = \mathrm{E}(Z_{0,n}^-)$. Furthermore, define $\xi_{i,n}^{(m),+} = Z_{i,n}^+ - K^+$ and $\xi_{i,n}^{(m),-} = Z_{i,n}^- - K^-$. Then, using the fact that $K^+ - K^- = 0$, we can write

$$\xi_{i,n}^{(m)} = Z_{i,n}^+ - Z_{i,n}^- = Z_{i,n}^+ - K^+ - (Z_{i,n}^- - K^-) = \xi_{i,n}^{(m),+} - \xi_{i,n}^{(m),-}.$$

Let $\mathbb{B}_n^{(m),+}$ and $\mathbb{B}_n^{(m),-}$ be the analogues of $\mathbb{B}_n^{(m)}$ defined from the sequences $(\xi_{i,n}^{(m),+})_{i\in\mathbb{Z}}$ and $(\xi_{i,n}^{(m),-})_{i\in\mathbb{Z}}$, respectively, and similarly for $\mathring{\mathbb{B}}_n^{(m),+}$ and $\mathring{\mathbb{B}}_n^{(m),-}$. The case treated above yields

$$\sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left|\check{\mathbb{B}}_n^{(m),+}(s,t,\boldsymbol{u}) - \mathring{\mathbb{B}}_n^{(m),+}(s,t,\boldsymbol{H}^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}))\right| \overset{\mathrm{P}}{\to} 0,$$
$$\sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left|\check{\mathbb{B}}_n^{(m),-}(s,t,\boldsymbol{u}) - \mathring{\mathbb{B}}_n^{(m),-}(s,t,\boldsymbol{H}^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}))\right| \overset{\mathrm{P}}{\to} 0.$$

The desired result finally follows from the fact that $\mathbb{B}_n^{(m)} = \mathbb{B}_n^{(m),+} - \mathbb{B}_n^{(m),-}$ and $\mathring{\mathbb{B}}_n^{(m)} = \mathring{\mathbb{B}}_n^{(m),+} - \mathring{\mathbb{B}}_n^{(m),-}$.

■

C On the set-up of the simulation experiments

For the numerical experiments involving serially dependent observations, we restricted ourselves to the bivariate case and only focused on the tests based on $\check{S}_n$ and $\hat{S}_n$. Given a bivariate copula C, two models were used to generate serially dependent observations under H_0 defined in (1.1).

- The first one is a simple autoregressive model of order one, AR(1). Let $\boldsymbol{U}_i$, $i \in \{-100,\dots,0,\dots,n\}$, be a bivariate i.i.d. sample from a copula C. Then, set $\boldsymbol{\epsilon}_i = (\Phi^{-1}(U_{i1}), \Phi^{-1}(U_{i2}))$, where Φ is the c.d.f. of the standard normal distribution, and $\boldsymbol{X}_{-100} = \boldsymbol{\epsilon}_{-100}$. Finally, for any $j \in \{1,2\}$ and $i \in \{-99,\dots,0,\dots,n\}$, compute recursively
$$X_{ij} = 0.5X_{i-1,j} + \epsilon_{ij}. \tag{AR1}$$
- The second model is a bivariate version of the exponential autoregressive (EXPAR) model considered in Auestad and Tjøstheim (1990) and Paparoditis and Politis (2001, Section 3.3) (see also Bücher and Kojadinovic, 2013). The sample $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n$ is generated as previously with (AR1) replaced by
$$X_{ij} = \{0.8 - 1.1\exp(-50X^2_{i-1,j})\}X_{i-1,j} + 0.1\epsilon_{ij}. \tag{EXPAR}$$

Data under $(\neg H_0)\cap H_{0,m}$, with $H_{0,m}$ as in (1.2), were generated using the procedures described above except that the bivariate random vectors $\boldsymbol{U}_i$, $i \in \{-100,\dots,0,\dots,n\}$ are independent such that $\boldsymbol{U}_i$, $i \in \{-100,\dots,0,\dots,k^\star\}$ are i.i.d. from a copula C_1 and $\boldsymbol{U}_i$, $i \in \{k^\star+1,\dots,n\}$ are i.i.d. from a copula C_2, where $C_1 \neq C_2$ and $k^\star = \lfloor nt\rfloor$ for some $t \in (0,1)$. The resulting samples $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n$ are therefore not samples under $H_{1,c}\cap H_{0,m}$ since the change in the dependence is gradual by (AR1) or (EXPAR). The copulas C_1 and C_2 were taken to be both either bivariate Clayton, Gumbel–Hougaard, Normal or Frank copulas such that C_1 has a Kendall's tau of 0.2 and C_2 a Kendall's tau of $\tau \in \{0.4, 0.6\}$. The parameter t defining $k^\star$ was chosen in $\{0.25, 0.5\}$.

The dependent multiplier sequences necessary to carry out the tests were generated using the "moving average approach" proposed initially in Bühlmann (1993, Section 6.2) and revisited in some detail in Bücher and Kojadinovic (2013, Section 6.1). A standard normal sequence was used for the required initial i.i.d. sequence. The kernel function κ in that procedure was chosen to be the Parzen kernel defined by $\kappa_P(x) = (1 - 6x^2 + 6|x|^3)\mathbf{1}(|x| \leq 1/2) + 2(1 - |x|)^3\mathbf{1}(1/2 < |x| \leq 1)$, $x \in \mathbb{R}$, which amounts to choosing the function φ in Condition (M3) as $x \mapsto (\kappa_P \star \kappa_P)(2x)/(\kappa_P \star \kappa_P)(0)$, where '$\star$' denotes the convolution operator. The value of the bandwidth parameter ℓ_n defined in Condition (M2) was chosen using the procedure described in Bücher and Kojadinovic (2013, Section 5). Two choices for the "combining" function ψ in that procedure were considered: the median and the maximum. Both choices led to similar rejection rates. The results reported in Tables 2 and 6 below are those obtained with ψ = maximum.

D Selected results of the simulation study

Tables 1 up to 6 provide partial results of the large-scale Monte Carlo simulation experiment described in Section 5. All the tests were carried out at the 5% level of significance.

Table 1: Percentage of rejection of H_0 computed from 1000 random samples of size $n \in \{50, 100, 200\}$ generated under H_0, where C is either the d-dimensional Clayton (Cl), the Gumbel–Hougaard (GH) or the normal (N) copula whose bivariate margins have a Kendall's tau of τ.

			Cl			GH			N		
d	n	τ	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R
2	50	0.00	6.2	4.0	4.6	5.4	2.9	4.5	7.3	3.4	4.8
		0.25	6.7	6.2	5.6	5.5	3.3	5.4	4.4	3.0	6.3
		0.50	5.6	7.9	6.0	4.4	3.3	4.6	4.4	5.3	4.9
		0.75	6.0	16.6	5.5	3.2	6.7	4.3	3.6	9.1	4.9
	100	0.00	4.9	3.5	5.3	5.2	4.1	5.5	4.3	2.8	5.5
		0.25	6.1	6.6	5.0	5.0	3.3	6.2	5.3	4.0	5.5
		0.50	4.4	9.3	5.9	3.7	2.8	5.7	3.1	3.4	5.3
		0.75	2.7	10.0	4.6	2.5	4.7	4.4	2.1	6.0	5.6
	200	0.00	4.0	3.5	5.2	4.3	4.0	5.2	5.4	4.9	4.3
		0.25	4.7	5.2	6.3	3.3	3.0	3.8	4.0	3.9	5.2
		0.50	5.1	8.5	4.9	3.2	2.3	4.5	4.0	4.7	4.8
		0.75	2.6	9.3	5.9	1.5	3.1	5.2	1.9	4.8	5.7
3	50	0.00	4.3	1.5	3.0	4.2	2.1	3.6	5.5	2.8	3.4
		0.25	6.3	5.0	5.1	5.5	1.0	5.1	5.3	3.0	4.3
		0.50	8.2	9.1	5.9	2.7	0.9	5.7	3.0	2.2	4.6
		0.75	2.0	2.9	6.9	0.5	0.4	6.3	1.1	1.3	4.1
	100	0.00	4.5	3.4	4.5	4.5	2.8	4.6	4.5	2.7	3.9
		0.25	5.0	5.1	5.4	4.2	2.6	4.4	5.4	3.5	4.5
		0.50	5.7	7.6	6.3	3.3	1.3	5.0	3.2	3.1	3.9
		0.75	2.5	4.9	5.0	1.0	1.0	5.2	0.8	1.6	5.5
	200	0.00	3.3	2.5	4.3	3.5	3.2	4.3	4.8	4.0	4.7
		0.25	6.6	7.1	5.5	5.0	3.3	4.5	4.8	4.1	5.0
		0.50	6.0	9.2	4.5	3.0	2.4	5.9	4.8	4.3	4.8
		0.75	2.9	6.4	6.4	0.7	0.9	3.8	1.3	2.2	4.9

Table 2: Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{100, 200\}$ generated under H_0 as explained in Appendix C, where C is either the bivariate Clayton (Cl), the Gumbel–Hougaard (GH), the normal (N) or the Frank (F) copula with a Kendall's tau of τ. The columns $\overline{\hat{\ell}_n^{opt}}$ and std give the mean and the standard deviation of the values of ℓ_n used for creating the dependent multiplier sequences.

			AR1				EXPAR			
C	n	τ	$\overline{\hat{\ell}_n^{opt}}$	std	$\check{S}_n$	$\hat{S}_n$	$\overline{\hat{\ell}_n^{opt}}$	std	$\check{S}_n$	$\hat{S}_n$
Cl	100	0.00	14.2	8.5	4.2	0.7	16.7	9.5	5.1	0.6
		0.25	14.1	8.6	5.9	1.7	16.7	10.2	5.5	2.3
		0.50	14.0	10.3	4.5	2.9	16.4	10.7	6.1	3.2
		0.75	13.3	10.0	2.7	3.3	15.5	10.7	5.1	4.7
	200	0.00	16.7	8.0	5.1	2.6	20.9	9.5	4.5	1.7
		0.25	16.0	7.3	5.1	3.0	20.5	9.7	4.1	2.0
		0.50	15.8	7.8	2.6	2.5	19.8	9.9	3.5	2.8
		0.75	15.5	9.0	1.6	3.6	19.0	9.0	4.3	4.7
GH	100	0.00	14.5	9.7	4.6	0.9	16.9	8.2	4.4	0.6
		0.25	14.1	8.7	4.9	1.5	17.1	10.2	5.0	0.6
		0.50	14.0	9.5	3.9	1.2	15.8	9.5	4.0	0.3
		0.75	13.7	10.0	2.6	0.5	15.2	9.2	1.6	0.4
	200	0.00	16.8	7.9	4.3	1.8	21.5	10.1	3.6	1.5
		0.25	16.6	8.8	5.5	2.0	20.9	11.5	5.1	1.1
		0.50	15.9	7.4	3.7	1.6	20.1	10.9	2.9	0.7
		0.75	15.5	8.7	1.3	0.9	18.8	9.1	1.6	0.2
N	100	0.00	14.1	7.9	5.0	1.1	17.3	9.2	5.4	1.4
		0.25	13.5	8.1	5.9	1.4	17.3	10.8	5.0	1.1
		0.50	13.5	9.1	3.3	1.4	16.4	9.7	4.5	1.0
		0.75	12.9	7.9	1.7	1.7	15.7	10.8	2.7	1.1
	200	0.00	16.3	6.2	5.4	1.9	20.7	8.7	3.7	1.5
		0.25	16.0	7.1	4.2	2.4	20.9	8.9	5.0	1.5
		0.50	16.1	7.8	4.2	3.2	19.8	10.4	2.9	1.8
		0.75	15.4	7.5	0.9	1.4	19.3	10.7	0.8	0.5
F	100	0.00	13.8	7.8	5.8	1.5	17.4	9.7	6.4	0.8
		0.25	14.2	9.3	5.5	1.3	16.6	9.7	4.8	0.5
		0.50	13.9	9.0	3.3	1.7	16.8	11.4	3.6	0.5
		0.75	13.5	9.7	1.5	0.6	15.8	10.3	3.1	1.2
	200	0.00	16.8	7.2	4.2	2.3	20.9	9.4	4.3	1.4
		0.25	16.0	7.0	6.0	2.9	20.6	8.7	3.6	1.1
		0.50	16.1	8.2	3.1	1.5	20.3	10.4	3.2	1.2
		0.75	15.8	8.8	0.9	0.5	19.6	9.0	1.2	0.8

Table 3: Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{50, 100, 200\}$ generated under $H_{0,m} \cap H_{1,c}$, where $H_{1,c}$ is defined in (5.1), $k^\star = \lfloor nt \rfloor$, C_1 and C_2 are both either bivariate Clayton (Cl), Gumbel–Hougaard (GH) or normal (N) copulas such that C_1 has a Kendall's tau of 0.2 and C_2 a Kendall's tau of τ.

			Cl			GH			N		
n	τ	t	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R
50	0.4	0.10	7.5	8.1	5.7	6.1	3.7	4.3	6.1	4.8	4.3
		0.25	12.1	10.6	4.0	9.4	5.1	5.0	10.4	7.6	5.5
		0.50	18.0	16.1	6.3	12.1	7.8	4.8	12.3	8.4	5.8
	0.6	0.10	14.5	17.0	5.4	11.4	7.7	6.2	11.4	9.8	7.2
		0.25	35.5	34.4	7.4	29.9	21.4	6.4	31.3	21.4	7.1
		0.50	47.3	41.6	7.0	45.3	30.3	8.9	46.0	33.9	9.1
100	0.4	0.10	7.1	8.7	6.1	6.6	5.1	5.1	5.8	5.2	5.3
		0.25	18.8	19.9	5.2	16.9	13.2	5.8	14.9	12.5	6.1
		0.50	26.5	26.4	7.3	23.8	18.8	7.3	22.6	19.1	7.9
	0.6	0.10	21.5	25.1	5.8	16.7	12.0	5.1	17.5	16.9	6.1
		0.25	65.1	66.0	6.3	61.2	51.5	7.5	62.9	54.8	9.9
		0.50	82.1	81.6	14.7	78.8	69.7	14.9	79.1	73.7	11.8
200	0.4	0.10	11.1	13.8	5.8	8.3	8.1	5.5	9.5	9.8	5.0
		0.25	30.8	33.9	5.9	27.6	24.8	6.4	29.6	28.3	6.8
		0.50	47.1	48.6	9.0	45.8	41.4	8.7	47.1	46.1	9.4
	0.6	0.10	36.4	41.3	6.8	34.4	31.7	7.1	36.0	36.3	6.7
		0.25	92.6	93.2	12.3	91.4	88.9	16.7	91.3	90.2	12.0
		0.50	98.9	99.3	22.2	98.5	98.1	22.0	99.3	99.1	21.1

Table 4: Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{100, 200\}$ generated under $H_{0,m} \cap H_{1,c}$, where $H_{1,c}$ is defined in (5.1), $k^\star = \lfloor nt \rfloor$, C_1 (resp. C_2) is a d-dimensional Clayton (resp. Gumbel–Hougaard) copula whose bivariate margins have a Kendall's tau of τ.

			$d=2$			$d=3$		
n	τ	t	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R
100	0.25	0.25	5.7	4.1	5.3	5.3	3.1	4.4
		0.50	6.2	5.7	5.6	9.1	6.0	5.8
		0.75	6.6	6.3	3.5	5.8	5.4	4.9
	0.50	0.25	5.5	5.9	5.8	4.6	2.9	5.0
		0.50	10.5	12.2	5.1	15.1	15.1	6.8
		0.75	8.3	11.9	4.4	7.7	9.9	5.3
	0.75	0.25	4.0	7.6	5.1	2.5	1.9	4.3
		0.50	12.5	19.9	6.0	9.8	13.2	5.2
		0.75	8.2	16.4	6.5	4.8	6.7	3.8
200	0.25	0.25	5.8	5.3	6.1	5.8	3.7	5.8
		0.50	8.5	8.7	5.4	9.8	9.7	6.6
		0.75	6.4	6.9	5.5	9.0	9.1	5.3
	0.50	0.25	10.4	11.5	6.8	12.6	10.1	6.7
		0.50	30.9	37.5	5.3	44.0	45.8	7.1
		0.75	16.3	23.2	6.1	20.1	27.0	5.1
	0.75	0.25	11.3	18.0	4.9	15.6	16.7	7.3
		0.50	43.4	54.4	6.2	58.9	63.1	5.2
		0.75	21.3	36.4	4.6	23.6	36.0	6.8

Table 5: Rejection percentage of H_0 computed from 1000 samples of size $n \in \{50, 100, 200\}$ such that the $\lfloor nt_1 \rfloor$ first observations of each sample are from a d-variate c.d.f. with normal copula and $N(0,1)$ margins (that is, from a multivariate standard normal c.d.f.), and the $n - \lfloor nt_1 \rfloor$ last observations are from a d-variate c.d.f. with normal copula whose first margin is the $N(\mu, 1)$ and whose $d-1$ remaining margins are the $N(0,1)$. The bivariate margins of the normal copula have a Kendall's tau of τ.

	$(\mu, t_1) =$		$(0.5, 0.25)$			$(0.5, 0.5)$			$(2, 0.25)$			$(2, 0.5)$		
d	n	τ	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R
2	50	0.00	6.7	3.8	9.0	6.5	3.1	17.7	4.6	2.4	70.1	5.1	2.6	98.5
		0.25	5.6	2.8	8.1	4.8	3.6	15.0	6.0	3.1	57.7	4.7	1.6	98.5
		0.50	4.4	4.1	9.6	3.4	2.9	13.0	18.4	5.1	39.6	5.7	0.9	99.3
	100	0.00	5.3	4.0	19.0	6.3	5.2	30.4	6.1	4.1	99.0	5.5	2.5	100.0
		0.25	5.0	4.4	13.9	3.7	2.3	24.9	8.6	4.7	97.5	4.4	2.2	100.0
		0.50	4.0	3.8	11.5	3.1	3.1	22.6	29.5	13.1	91.6	12.4	2.0	100.0
	200	0.00	5.5	5.2	34.1	3.9	3.4	61.9	4.3	3.3	100.0	5.6	4.1	100.0
		0.25	3.9	3.4	27.6	4.1	3.5	51.3	13.6	8.9	100.0	8.1	4.0	100.0
		0.50	3.4	3.9	18.9	2.8	2.9	43.1	57.8	39.4	100.0	34.8	8.3	100.0
3	50	0.00	4.9	1.6	5.0	4.5	1.7	10.0	4.8	1.9	36.5	6.0	2.3	79.6
		0.25	5.0	2.7	6.9	5.0	2.6	9.9	6.9	2.6	24.8	5.1	2.6	87.4
		0.50	3.7	2.6	5.5	3.3	1.3	8.9	9.4	3.5	17.4	3.0	0.7	94.2
	100	0.00	4.5	2.3	11.3	4.2	2.5	18.6	4.8	3.0	87.5	3.6	2.0	99.6
		0.25	4.9	3.1	10.7	5.1	3.5	14.8	6.7	3.8	67.9	5.9	4.2	99.9
		0.50	2.8	2.0	7.3	3.0	2.4	13.7	16.9	8.6	60.6	6.0	1.3	100.0
	200	0.00	3.0	2.4	20.1	4.5	4.0	37.3	5.3	3.3	100.0	4.3	3.2	100.0
		0.25	4.8	4.1	15.3	5.6	4.4	30.6	11.9	8.1	99.2	7.4	5.2	100.0
		0.50	4.9	3.7	11.8	3.8	3.2	24.9	41.0	30.9	99.2	23.9	7.8	100.0

Table 6: Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{100, 200\}$ generated under $\neg H_0$ as explained in the second paragraph of Appendix C, where C_1 and C_2 are both bivariate Gumbel–Hougaard copulas such that C_1 has a Kendall's tau of 0.2 and C_2 a Kendall's tau of τ. The columns $\overline{\hat{\ell}_n^{opt}}$ and std give the mean and the standard deviation of the values of ℓ_n used for creating the dependent multiplier sequences.

			AR1				EXPAR			
n	t	τ	$\overline{\hat{\ell}_n^{opt}}$	std	$\check{S}_n$	$\hat{S}_n$	$\overline{\hat{\ell}_n^{opt}}$	std	$\check{S}_n$	$\hat{S}_n$
100	0.25	0.4	14.2	10.1	13.8	3.8	16.9	10.5	10.6	1.5
		0.6	14.1	9.1	39.9	10.6	15.8	9.5	32.6	8.5
	0.50	0.4	14.3	10.1	18.0	5.3	17.1	10.3	13.5	2.5
		0.6	13.9	8.6	57.2	25.2	16.5	9.3	54.8	17.1
200	0.25	0.4	16.5	8.6	16.4	8.6	20.6	9.8	14.6	4.5
		0.6	16.4	8.0	71.2	46.0	19.6	9.0	63.1	32.5
	0.50	0.4	16.6	7.4	31.9	19.0	20.7	8.8	25.8	12.4
		0.6	16.3	7.1	89.8	75.5	20.6	9.1	80.9	62.5

Chapitre 4

Tests de détection de rupture dans le rho de Spearman multivarié

Sommaire

4.1 Introduction

Dans ce chapitre, nous présentons une seconde classe de tests pour la détection de rupture dans la copule d'observations multivariées, qui se focalise sur la détection d'un changement dans le rho de Spearman des observations. Nous considérerons en particulier les versions multivariées du rho de Spearman proposées dans Schmid et Schmidt (2007a).

Les tests étudiés reposent sur une approche similaire à l'approche CUSUM. Trois statistiques de tests sont considérées :

$$S_{n,i} = \sup_{s\in[0,1]} \sqrt{n}\lambda_n(0,s)\lambda_n(s,1) \left|\rho_i(C_{1:\lfloor ns\rfloor}) - \rho_i(C_{\lfloor ns\rfloor+1:n,i})\right|, \quad i = 1,2,3,$$

où ρ_1 (resp. ρ_2 et ρ_3) est défini dans (1.11) (resp. dans (1.13) et (1.15)), et où $C_{1:\lfloor ns\rfloor}$ et $C_{\lfloor ns\rfloor+1:n}$ sont définies dans (2.10). Il est intéressant de remarquer que, par linéarité de l'intégrale intervenant dans la définition de ρ_i, $S_{n,i}$ peut se réécrire comme

$$S_{n,i} = \sup_{s\in[0,1]} \left|\rho_i(\mathbb{D}_n(s,.))\right|,$$

où $\mathbb{D}_n$ est le processus défini dans l'équation (3.4) du chapitre précédent. Ainsi, sous réserve que les conditions nécéssaires sur les dérivées partielles soient satisfaites, l'étude asymptotique du test et la validité de sa mise en œuvre sous $\mathcal{H}_0$ définie dans (3.1), découlent immédiatement du chapitre précédent.

L'une des contributions de l'article ci-après est de montrer que les conditions mentionnées précédemment ne sont pas nécessaires pour l'étude asymptotique sous l'hypothèse $\mathcal{H}_0$. En cela, la catégorie de tests étudiée est plus générale que le test développé dans le chapitre précédent. L'inconvénient principal des procédures statistiques proposées est que, par construction, elles n'auront aucune puissance pour des alternatives contenant des ruptures dans la copule à valeur constante du rho de Spearman.

4.2 Un avant-propos

Dans cette section, au travers du lemme suivant, nous allons voir que pour $i = 1, 2, 3$, l'écriture de $\rho_i\{\mathbb{C}_C(s, t, .)\}$, $(s, t) \in \Delta$, où $\mathbb{C}_C$ est défini dans (2.8), ne fait pas intervenir les dérivées partielles $\dot{C}_j$, $j \in \{1, \dots, d\}$, de la copule C. Ce résultat donne l'intuition que la condition 2.2.1 portant sur les dérivées partielles de la copule n'est pas nécessaire pour étudier le comportement asymptotique de la statistique $S_{n,i}$ pour $i = 1, 2, 3$.

Lemme 4.2.1. *Pour tout $j \in \{1, \dots, d\}$,*

$$\int_{[0,1]^d} \dot{C}_j(\boldsymbol{u})\, \mathbb{B}_C(0, 1, \boldsymbol{u}^{\{j\}}) \mathrm{d}\boldsymbol{u} = \mathbb{E}_C\Big\{ \prod_{\substack{q=1 \\ q \neq j}}^{d} (1 - U_q) \mathbb{B}_C(0, 1, \boldsymbol{U}^{\{j\}}) \Big\},$$

où $\boldsymbol{U} = (U_1, \dots, U_d)$ est un vecteur aléatoire de f.d.r. C.

Démonstration. Soit $j \in \{1, \dots, d\}$ fixé. D'une part la copule C peut s'écrire pour $\boldsymbol{u} \in [0, 1]^d$ de la façon suivante :

$$C(\boldsymbol{u}) = \int_0^{u_1} \dots \int_0^{u_d} \mathrm{d}C(\boldsymbol{v}) = \mathbb{E}_C\Big\{ \prod_{q=1}^{d} \mathbf{1}(U_q \leq u_q) \Big\}.$$

La dérivée partielle $\dot{C}_j$ de C peut s'écrire quant à elle pour $\boldsymbol{u} \in [0, 1]^d$ sous la forme d'espérance conditionnelle :

$$\dot{C}_j(\boldsymbol{u}) = \mathbb{E}_C\Big\{ \prod_{\substack{q=1 \\ q \neq j}}^{d} \mathbf{1}(U_q \leq u_q) | U_j = u_j \Big\}.$$

Par conséquent,

$$\int_{[0,1]^d} \dot{C}_j(\boldsymbol{u})\, \mathbb{B}_C(0, 1, \boldsymbol{u}^{\{j\}}) \mathrm{d}\boldsymbol{u}$$

$$= \int_{[0,1]^d} \mathbb{B}_C(0, 1, \boldsymbol{u}^{\{j\}}) \mathbb{E}_C\Big\{ \prod_{\substack{q=1 \\ q \neq j}}^{d} \mathbf{1}(U_q \leq u_q) | U_j = u_j \Big\} \mathrm{d}\boldsymbol{u}$$

$$= \int_0^1 \int_{[0,1]^{d-1}} \mathbb{E}_C \Big\{ \mathbb{B}_C(0,1,\boldsymbol{U}^{\{j\}}) \prod_{\substack{q=1 \\ q \neq j}}^{d} \mathbf{1}(U_q \leq u_q) | U_j = u_j \Big\} \mathrm{d}\boldsymbol{u}^{D\setminus\{j\}} \mathrm{d}u_j,$$

où le vecteur $\boldsymbol{u}^{D\setminus\{j\}}$ est défini dans (1.1). Ainsi en utilisant le théorème de Fubini, la dernière égalité peut s'écrire

$$\int_0^1 \mathbb{E}_C \Big\{ \int_{[0,1]^{d-1}} \mathbb{B}_C(0,1,\boldsymbol{U}^{\{j\}}) \prod_{\substack{q=1 \\ q \neq j}}^{d} \mathbf{1}(U_q \leq u_q) \mathrm{d}\boldsymbol{u}^{D\setminus\{j\}} | U_j = u_j \Big\} \mathrm{d}u_j$$

$$= \mathbb{E}_C \Big\{ \int_{[0,1]^{d-1}} \mathbb{B}_C(0,1,\boldsymbol{U}^{\{j\}}) \prod_{\substack{q=1 \\ q \neq j}}^{d} \mathbf{1}(U_q \leq u_q) \mathrm{d}\boldsymbol{u}^{D\setminus\{j\}} \Big\}$$

$$= \mathbb{E}_C \Big\{ \prod_{\substack{q=1 \\ q \neq j}}^{d} (1 - U_q) \mathbb{B}_C(0,1,\boldsymbol{U}^{\{j\}}) \Big\}.$$

$\square$

Testing the constancy of Spearman's rho in multivariate time series

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Abstract

A class of tests for change-point detection designed to be particularly sensitive to changes in the cross-sectional rank correlation of multivariate time series is proposed. The derived procedures are based on several multivariate extensions of Spearman's rho. Two approaches to carry out the tests are studied: the first one is based on resampling, the second one consists of estimating the asymptotic null distribution. The asymptotic validity of both techniques is proved under the null for strongly mixing observations. A procedure for estimating a key bandwidth parameter involved in both approaches is proposed, making the derived tests parameter-free. Their finite-sample behavior is investigated through Monte Carlo experiments. Practical recommendations are made and an illustration on trivariate financial data is finally presented.

1 Introduction

Given a multivariate times series $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ of d-dimensional observations, we aim at testing

$$H_0 : \exists\, F \text{ such that } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n \text{ have c.d.f. } F \tag{1.1}$$

against $\neg H_0$. Such statistical procedures are commonly referred to as *tests for change-point detection* (see e.g. Csörgő and Horváth, 1997, for an overview of possible approaches). The majority of tests for H_0 developed in the literature deal with the case $d = 1$. We aim at developing *nonparametric* tests for *multivariate* time series that are particularly sensitive to changes in the *dependence* among the components of the d-dimensional observations. The availability of such tests seems to be of great practical importance for the analysis of economic data, among others. In particular, assessing whether the dependence among financial assets can be considered constant or not over a given time period appears crucial for risk management, portfolio optimization and related statistical modeling (see e.g. Wied et al., 2014; Dehling et al., 2014, and the references therein for a more detailed discussion about the motivation for such statistical procedures).

The above context, rather naturally, suggests to address the informal notion of *dependence* through that of *copula* (see e.g. Nelsen, 2006). Assume that H_0 holds and that, additionally, the common marginal c.d.f.s $F_1, \dots, F_d$ of $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are continuous. Then, from the work of Sklar (1959), the common multivariate c.d.f. F of the observations can be written as

$$F(\boldsymbol{x}) = C\{F_1(x_1), \dots, F_d(x_d)\}, \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

where the function $C : [0,1]^d \to [0,1]$ is the unique *copula* associated with F. It follows that H_0 can be rewritten as $H_{0,m} \cap H_{0,c}$, where

$$H_{0,m} : \exists\, F_1, \dots, F_d \text{ such that } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n \text{ have marginal c.d.f.s } F_1, \dots, F_d, \tag{1.2}$$

$$H_{0,c} : \exists\, C \text{ such that } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n \text{ have copula } C. \tag{1.3}$$

Several nonparametric tests designed to be particularly sensitive to certain alternatives under $H_{0,m} \cap \neg H_{0,c}$ have been proposed in the literature. Tests for the constancy of Kendall's tau (which is a functional of C) were investigated by Gombay and Horváth (1999) (see also Gombay and Horváth, 2002) and Quessy et al. (2013) in the case of serially independent observations. A version of the previous tests adapted to a very general class of bivariate time series was proposed by Dehling et al. (2014). Recent multivariate alternatives are the tests studied in Bücher et al. (2014, see also the references therein) based on Cramér–von Mises functionals of the *sequential empirical copula process*.

The aim of this work is to derive tests for the constancy of several multivariate extensions of Spearman's rho (which are also functionals of C) in multivariate strongly mixing time series. A similar problem was recently tackled by Wied et al. (2014). However, as the functional they considered does not exactly correspond to a multivariate extension of Spearman's rho (because of the way ranks are calculated), the corresponding test turn out to have a rather low power. We remedy to that situation by computing ranks with respect to the relevant subsamples. From a theoretical perspective, as in Wied et al. (2014), no assumptions on the first order partial derivatives of the copula are made. The latter is actually an advantage of the studied tests over that investigated in Bücher et al. (2014). An inconvenience with respect to the aforementioned approach is however that, as all tests based on moments of copulas (such as Spearman's rho or Kendall's tau), the derived tests will have no power, by construction, against alternatives involving changes in the copula at a constant value of Spearman's rho.

To carry out the tests, we propose two approaches to compute approximate p-values: the first one is based on resampling while the second one consists of estimating the asymptotic null distribution. In addition, a procedure to estimate a key bandwidth parameter involved in both approaches is proposed, making the derived tests fully data-driven. The versions of the studied tests based on the estimation of the asymptotic null distribution can be seen as alternatives to the test based on Kendall's tau recently proposed by Dehling et al. (2014).

The paper is organized as follows. The test statistics are defined in the second section and their limiting null distribution is established under strong mixing. Section 3 presents two approaches to compute approximate p-values based, respectively, on bootstrapping and on the estimation of an asymptotic variance. The fourth section partially reports the results of Monte Carlo experiments involving bivariate and fourvariate time series. The last section contains practical recommendations and an illustration on trivariate financial data.

In the rest of the paper, the arrow '$\leadsto$' denotes weak convergence in the sense of Definition 1.3.3 in van der Vaart and Wellner (2000). Also, given a set T, $\ell^\infty(T)$ denotes the space of all bounded real-valued functions on T equipped with the uniform metric.

2 Test statistics

2.1 Multivariate extensions of Spearman's rho and their estimation

Spearman's rho is a very well-known measure of bivariate dependence (see e.g. Nelsen, 2006, Section 5.1 and the references therein). For a bivariate random vector with continuous margins and copula C, it can be expressed as

$$\rho(C) = 12 \int_{[0,1]^2} C(\boldsymbol{u}) \mathrm{d}\boldsymbol{u} - 3 = 12 \int_{[0,1]^2} u_1 u_2 \mathrm{d}C(\boldsymbol{u}) - 3.$$

When the random vector of interest is d-dimensional with $d > 2$, the following three possible extensions were proposed by Schmid and Schmidt (2007):

$$\begin{aligned} \rho_1(C) &= \frac{d+1}{2^d - d - 1} \left\{ 2^d \int_{[0,1]^d} C(\boldsymbol{u}) \mathrm{d}\boldsymbol{u} - 1 \right\}, \\ \rho_2(C) &= \rho_1(\bar{C}), \\ \rho_3(C) &= \binom{d}{2}^{-1} \sum_{1 \leq i < j \leq d} \rho(C^{(i,j)}), \end{aligned}$$

where $C^{(i,j)}$ is the bivariate margin obtained from C by keeping dimensions i and j, and $\bar{C}$ is the survival function corresponding to C. It is well-known that the latter can be expressed in terms of C. To see this, let $D = \{1, \dots, d\}$ and, for any $\boldsymbol{u} \in [0,1]^d$ and $A \subseteq D$, let $\boldsymbol{u}^A$ be the vector of $[0,1]^d$ such that $u_i^A = u_i$ if $i \in A$ and $u_i^A = 1$ otherwise.

Then, for any $\boldsymbol{u} \in [0,1]^d$, $\bar{C}(\boldsymbol{u}) = \sum_{A \subseteq D} (-1)^{|A|} C(\boldsymbol{u}^A)$. Other related d-dimensional coefficients are considered in Quessy (2009).

Let us now discuss the estimation of the above theoretical quantities. Specifically, we assume that we have at hand n copies $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ of a d-dimensional random vector $\boldsymbol{X}$ with copula C and continuous margins. Given an estimator of C, natural estimators of $\rho_1(C)$, $\rho_2(C)$ and $\rho_3(C)$ can be obtained using the plug-in principle. Restricting attention to a sample $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$, $1 \leq k \leq l \leq n$, for reasons that will become clear in the next subsection, a natural estimator of C is given by

$$C_{k:l}(\boldsymbol{u}) = \frac{1}{l-k+1} \sum_{i=k}^{l} \mathbf{1}(\hat{\boldsymbol{U}}_i^{k:l} \leq \boldsymbol{u}), \qquad \boldsymbol{u} \in [0,1]^d, \tag{2.1}$$

where

$$\hat{\boldsymbol{U}}_i^{k:l} = \frac{1}{l-k+1}(R_{i1}^{k:l}, \ldots, R_{id}^{k:l}), \qquad i \in \{k, \ldots, l\}, \tag{2.2}$$

with $R_{ij}^{k:l} = \sum_{t=k}^{l} \mathbf{1}(X_{tj} \leq X_{ij})$ the maximal rank of X_{ij} among $X_{kj}, \ldots, X_{lj}$. The quantity given by (2.1) is commonly referred to as the *empirical copula* of $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$ (see e.g. Rüschendorf, 1976; Deheuvels, 1981). Corresponding natural estimators of the three aforementioned multivariate versions of Spearman's rho are therefore $\rho_1(C_{k:l})$, $\rho_2(C_{k:l})$ and $\rho_3(C_{k:l})$, respectively.

It is important to notice that we do not necessarily assume the observations to be serially independent. Serial independence *and* continuity of the marginal distributions together guarantee the absence of ties in the d component series. However, continuity of the marginal distributions alone is *not* sufficient to guarantee the absence of ties when the observations are serially dependent (see e.g. Bücher and Segers, 2014, Example 4.2). This is the reason why maximal ranks are used in (2.2) above. The possible presence of ties in the component series makes the study of the tests under consideration substantially more complicated.

2.2 Change-point statistics

To derive tests for change-point detection particularly sensitive to changes in the strength of the cross-sectional dependence, one natural possibility is to base these tests on differences of Spearman's rhos. By analogy with the classical approach to change-point analysis (see e.g. Csörgő and Horváth, 1997), one could for instance consider the following three test statistics:

$$S_{n,i} = \max_{1 \leq k \leq n-1} \frac{k(n-k)}{n^{3/2}} \left| \rho_i(C_{1:k}) - \rho_i(C_{k+1:n}) \right|, \qquad i \in \{1,2,3\}, \tag{2.3}$$

where $C_{1:k}$ and $C_{k+1:n}$ are the empirical copulas of the subsamples $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ and $\boldsymbol{X}_{k+1}, \ldots, \boldsymbol{X}_n$, respectively, defined analogously to (2.1). All three statistics above turn out to be particular cases of a generic statistic which is the primary focus of this work. Before we can define it, some additional notation is necessary.

For any $A \subseteq D$, let ϕ_A be the map from $\ell^\infty([0,1]^d)$ to $\mathbb{R}$ defined by

$$\phi_A(g) = \int_{[0,1]^d} g(\boldsymbol{u}^A) \mathrm{d}\boldsymbol{u}, \qquad g \in \ell^\infty([0,1]^d). \tag{2.4}$$

Then, define the empirical process

$$\mathbb{T}_{n,A}(s) = \sqrt{n}\,\lambda_n(0,s)\,\lambda_n(s,1)\,\{\phi_A(C_{1:\lfloor ns\rfloor}) - \phi_A(C_{\lfloor ns\rfloor+1:n})\}, \qquad s \in [0,1],$$

where $\lambda_n(s,t) = (\lfloor nt\rfloor - \lfloor ns\rfloor)/n$ for $(s,t) \in \Delta = \{(s,t) \in [0,1]^2 : s \leq t\}$, and with the additional convention that $C_{k:l} = 0$ whenever $k > l$. Simple calculations reveal that $\mathbb{T}_{n,\emptyset} = 0$. Next, consider the $2^d - 1$-dimensional vector of processes

$$\mathbb{T}_n = (\mathbb{T}_{n,\{1\}}, \mathbb{T}_{n,\{2\}}, \dots, \mathbb{T}_{n,D}). \tag{2.5}$$

Finally, given a function $f : \mathbb{R}^{2^d-1} \to \mathbb{R}$, define the generic change-point statistic

$$S_{n,f} = \sup_{s\in[0,1]} |f\{\mathbb{T}_n(s)\}| = \max_{1\leq k\leq n-1} |f\{\mathbb{T}_n(k/n)\}|. \tag{2.6}$$

We shall now verify that the statistics $S_{n,i}$, $i \in \{1,2,3\}$, given by (2.3) are particular cases of $S_{n,f}$ when f is linear, that is, when there exists a vector $\boldsymbol{a} \in \mathbb{R}^{2^d-1}$ such that, for any $\boldsymbol{x} \in \mathbb{R}^{2^d-1}$, $f(\boldsymbol{x}) = \boldsymbol{a}^\top \boldsymbol{x}$. As we continue, with some abuse of notation, we index the components of vectors of $\mathbb{R}^{2^d-1}$ by subsets of D of cardinality greater than 1, i.e., for any $\boldsymbol{x} \in \mathbb{R}^{2^d-1}$, we write $\boldsymbol{x} = (x_{\{1\}}, x_{\{2\}}, \dots, x_D)$. Then, we have $S_{n,i} = S_{n,f_i}$, $i \in \{1,2,3\}$, where, for any $\boldsymbol{x} \in \mathbb{R}^{2^d-1}$,

$$f_1(\boldsymbol{x}) = \frac{(d+1)2^d}{2^d-d-1} x_D, \quad f_2(\boldsymbol{x}) = \frac{(d+1)2^d}{2^d-d-1} \sum_{\substack{A\subseteq D\\|A|\geq 1}} (-1)^{|A|} x_A, \quad f_3(\boldsymbol{x}) = \frac{24}{d(d-1)} \sum_{\substack{A\subseteq D\\|A|=2}} x_A.$$

Similar relationships hold for the statistics constructed from the additional coefficients mentioned in Quessy (2009), though the corresponding functions f are not necessarily linear anymore but only continuous.

Let us make a brief remark concerning the statistic $S_{n,2}$. Expressing it as S_{n,f_2} above is clearly not the most efficient way to compute it. To see this, for any $1 \leq k \leq l \leq n$, define

$$\bar{C}_{k:l}(\boldsymbol{u}) = \frac{1}{l-k+1} \sum_{i=k}^{l} \mathbf{1}(\hat{\boldsymbol{U}}_i^{k:l} > \boldsymbol{u}), \qquad \boldsymbol{u} \in [0,1]^d,$$

where the $\hat{\boldsymbol{U}}_i^{k:l}$ are defined in (2.2), and notice that $\bar{C}_{k:l}(\boldsymbol{u}) = \sum_{A\subseteq D} (-1)^{|A|} C_{k:l}(\boldsymbol{u}^A)$, $\boldsymbol{u} \in [0,1]^d$, where $C_{k:l}$ is defined in (2.1). Then, by definition of ρ_2,

$$S_{n,2} = \max_{1\leq k\leq n-1} \frac{k(n-k)}{n^{3/2}} \left|\rho_1(\bar{C}_{1:k}) - \rho_1(\bar{C}_{k+1:n})\right|.$$

Under the assumption of no ties in the d component series, some additional simple calculations reveal that the latter is actually nothing else than $S_{n,1}$ computed from the sample $-\boldsymbol{X}_1, \dots, -\boldsymbol{X}_n$.

We end this section by a discussion of the differences between $S_{n,1}$ and the similar statistic considered in Wied et al. (2014). Instead of basing their approach on the empirical copula, these authors considered the alternative estimator of C defined, for any $1 \leq k \leq l \leq n$, as

$$C_{k:l,n}(\boldsymbol{u}) = \frac{1}{l-k+1} \sum_{i=k}^{l} \mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n} \leq \boldsymbol{u}), \qquad \boldsymbol{u} \in [0,1]^d, \tag{2.7}$$

with the convention that $C_{k:l,n} = 0$ if $k > l$. The apparently subtle yet crucial difference between $C_{k:l}$ in (2.1) and $C_{k:l,n}$ above is that the scaled ranks are computed relative to the complete sample $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ for $C_{k:l,n}$, while, for $C_{k:l}$, they are computed relative to the subsample $\boldsymbol{X}_k, \dots, \boldsymbol{X}_l$. As a consequence, the analogue of the statistic $S_{n,1}$ considered in Wied et al. (2014) is not really a maximally selected absolute difference of sample Spearman's rhos. From a practical perspective, as illustrated empirically in Bücher et al. (2014), the use of $C_{k:l}$ instead of $C_{k:l,n}$ in a change-point detection framework results in tests that are more powerful when the change in distribution in only due to a change in the copula. We provide similar empirical evidence in Section 4: tests based on $S_{n,1}$ appear substantially more powerful than their analogues based on (2.7) for alternatives involving a change of $\rho_1(C)$ at constant margins. Reasons that explain this improved efficiency are discussed in Bücher et al. (2014, Section 2).

2.3 Limiting null distribution under strong mixing

Let us first recall the notion of *strongly mixing sequence*. For a sequence of d-dimensional random vectors $(\boldsymbol{Y}_i)_{i\in\mathbb{Z}}$, the σ-field generated by $(\boldsymbol{Y}_i)_{a\leq i\leq b}$, $a, b \in \mathbb{Z} \cup \{-\infty, +\infty\}$, is denoted by $\mathcal{F}_a^b$. The strong mixing coefficients corresponding to the sequence $(\boldsymbol{Y}_i)_{i\in\mathbb{Z}}$ are defined by

$$\alpha_r = \sup_{p\in\mathbb{Z}} \sup_{A\in\mathcal{F}_{-\infty}^p, B\in\mathcal{F}_{p+r}^{+\infty}} |P(A\cap B) - P(A)P(B)|$$

for positive integer r. The sequence $(\boldsymbol{Y}_i)_{i\in\mathbb{Z}}$ is said to be *strongly mixing* if $\alpha_r \to 0$ as $r \to \infty$.

The limiting null distribution of the vector of empirical process $\mathbb{T}_n$ defined in (2.5) can be obtained by rewriting its components in terms of the processes

$$\mathbb{S}_{n,A}(s,t) = \sqrt{n}\lambda_n(s,t)\{\phi_A(C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}) - \phi_A(C)\}, \qquad (s,t) \in \Delta, \tag{2.8}$$

for $A \subseteq D, |A| \geq 1$. Indeed, it easy to verify that, under H_0 defined in (1.1),

$$\mathbb{T}_{n,A}(s) = \lambda_n(s,1)\mathbb{S}_{n,A}(0,s) - \lambda_n(0,s)\mathbb{S}_{n,A}(s,1), \qquad s \in [0,1]. \tag{2.9}$$

As we shall see below, the limiting null distribution of $\mathbb{T}_n$ is then a mere consequence of the fact that the empirical processes $\mathbb{S}_{n,A}$, $A \subseteq D$, $|A| \geq 1$, are asymptotically equivalent to continuous functionals of the sequential empirical process

$$\mathbb{B}_n(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \{\mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}) - C(\boldsymbol{u})\}, \qquad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d, \tag{2.10}$$

where $\boldsymbol{U}_1, \dots, \boldsymbol{U}_n$ is the unobservable sample obtained from $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ by the probability integral transforms $U_{ij} = F_j(X_{ij})$, $i \in \{1, \dots, n\}$, $j \in D$.

If $\boldsymbol{U}_1, \dots, \boldsymbol{U}_n$ is drawn from a strictly stationary sequence $(\boldsymbol{U}_i)_{i\in\mathbb{Z}}$ whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$ with $a > 1$, we have from Bücher (2013) that $\mathbb{B}_n(0,\cdot,\cdot)$ converges weakly in $\ell^\infty([0,1]^{d+1})$ to a tight centered Gaussian process $\mathbb{B}_C^\circ$ with covariance function $\mathrm{cov}\{\mathbb{B}_C^\circ(s,\boldsymbol{u}), \mathbb{B}_C^\circ(t,\boldsymbol{v})\} = (s\wedge t)\kappa_C(\boldsymbol{u},\boldsymbol{v})$, $(s,\boldsymbol{u}),(t,\boldsymbol{v}) \in [0,1]^{d+1}$, where

$$\kappa_C(\boldsymbol{u},\boldsymbol{v}) = \mathrm{cov}\{\mathbb{B}_C^\circ(1,\boldsymbol{u}), \mathbb{B}_C^\circ(1,\boldsymbol{v})\} = \sum_{k\in\mathbb{Z}} \mathrm{cov}\{\mathbf{1}(\boldsymbol{U}_0 \leq \boldsymbol{u}), \mathbf{1}(\boldsymbol{U}_k \leq \boldsymbol{v})\}. \tag{2.11}$$

As a consequence of the continuous mapping theorem, $\mathbb{B}_n \rightsquigarrow \mathbb{B}_C$ in $\ell^\infty(\Delta \times [0,1]^d)$, where

$$\mathbb{B}_C(s,t,\boldsymbol{u}) = \mathbb{B}_C^\circ(t,\boldsymbol{u}) - \mathbb{B}_C^\circ(s,\boldsymbol{u}), \qquad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d. \tag{2.12}$$

The following proposition, proved in Appendix A, is the key step for obtaining the limiting null distribution of the vector of processes $\mathbb{T}_n$ defined in (2.5).

Proposition 2.1. *Assume that $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n$ is drawn from a strictly stationary sequence $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$ with continuous margins and whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$, $a > 1$. Then, for any $A \subseteq D$, $|A| \geq 1$,*

$$\sup_{(s,t)\in\Delta} |\mathbb{S}_{n,A}(s,t) - \psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\}| = o_{\mathrm{P}}(1), \tag{2.13}$$

where $\psi_{C,A}$ is a linear map from $\ell^\infty([0,1]^d)$ to $\mathbb{R}$ defined by

$$\psi_{C,A}(g) = \phi_A(g) - \int_{[0,1]^d} \sum_{j\in A} \prod_{l\in A\setminus\{j\}} (1-v_l) g(\boldsymbol{v}^{\{j\}}) \mathrm{d}C(\boldsymbol{v}), \qquad g \in \ell^\infty([0,1]^d), \tag{2.14}$$

with ϕ_A given in (2.4).

The next result, proved in Appendix B, is a consequence of the previous proposition and establishes the limiting null distribution of the generic statistic $S_{n,f}$ defined in (2.6) under strong mixing.

Corollary 2.2. *Under the conditions of Proposition 2.1,*

$$\mathbb{T}_n = \big(\mathbb{T}_{n,\{1\}}, \mathbb{T}_{n,\{2\}}, \dots, \mathbb{T}_{n,D}\big) \rightsquigarrow \mathbb{T}_C = \big(\mathbb{T}_{C,\{1\}}, \mathbb{T}_{C,\{2\}}, \dots, \mathbb{T}_{C,D}\big) \tag{2.15}$$

in $\{\ell^\infty([0,1])\}^{2^d-1}$, where

$$\mathbb{T}_C(s) = \psi_C\{\mathbb{B}_C(0,s,\cdot) - s\mathbb{B}_C(0,1,\cdot)\}, \qquad s \in [0,1], \tag{2.16}$$

with $\mathbb{B}_C$ defined in (2.12) *and ψ_C a map from $\ell^\infty([0,1]^d)$ to $\mathbb{R}^{2^d-1}$ defined by*

$$\psi_C(g) = \big(\psi_{C,\{1\}}(g), \psi_{C,\{2\}}(g), \dots, \psi_{C,D}(g)\big), \qquad g \in \ell^\infty([0,1]^d). \tag{2.17}$$

As a consequence, for any $f : \mathbb{R}^{2^d-1} \to \mathbb{R}$ continuous,

$$S_{n,f} = \sup_{s\in[0,1]} |f\{\mathbb{T}_n(s)\}| \rightsquigarrow S_{C,f} = \sup_{s\in[0,1]} |f\{\mathbb{T}_C(s)\}|,$$

and, if f is additionally linear and $\sigma_{C,f}^2 = \mathrm{var}[f \circ \psi_C\{\mathbb{B}_C(0,1,\cdot)\}] > 0$, the weak limit of $\sigma_{C,f}^{-1} S_{n,f}$ is equal in distribution to $\sup_{s\in[0,1]} |\mathbb{U}(s)|$, where $\mathbb{U}$ is a standard Brownian bridge on $[0,1]$.

3 Computation of approximate p-values

Corollary 2.2 suggests two related ways to compute p-values for the generic test statistic $S_{n,f}$ defined in (2.6). The first approach, based on resampling, consists of exploiting the fact that, under H_0, $\mathbb{T}_n$ defined in (2.5) is asymptotically equivalent to a continuous functional of the sequential empirical process $\mathbb{B}_n$ defined in (2.10) and can be applied as soon as $f : \mathbb{R}^{2^d-1} \to \mathbb{R}$ is continuous. The second approach, restricted to the situation when f is linear, is motivated by the last claim of Corollary 2.2. It consists of estimating $\sigma_{C,f}^2$ and thus the asymptotic null distribution of $S_{n,f}$.

3.1 Approximate p-values by bootstrapping

The first approach that we consider consists of bootstrapping the vector of empirical processes $\mathbb{T}_n$ defined in (2.5) using a bootstrap for the sequential empirical process $\mathbb{B}_n$. This way of proceeding actually allows us to consider not only linear but also *continuous* functions f in (2.6). More specifically, we consider a *multiplier bootstrap* for $\mathbb{B}_n$ in the spirit of van der Vaart and Wellner (2000, Chapter 2.9) when observations are serially independent, or Bühlmann (1993, Section 3.3) when they are serially dependent. In the latter case, we rely on the recent work of Bücher and Kojadinovic (2013).

The notion of *multiplier sequence* is central to this resampling technique. We say that a sequence of random variables $(\xi_{i,n})_{i\in\mathbb{Z}}$ is an *i.i.d. multiplier sequence* if:

(M0) $(\xi_{i,n})_{i\in\mathbb{Z}}$ is i.i.d., independent of $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n$, with distribution not changing with n, having mean 0, variance 1, and being such that $\int_0^\infty \{\mathrm{P}(|\xi_{0,n}|>x)\}^{1/2}\mathrm{d}x<\infty$.

We say that a sequence of random variables $(\xi_{i,n})_{i\in\mathbb{Z}}$ is a *dependent multiplier sequence* if:

(M1) The sequence $(\xi_{i,n})_{i\in\mathbb{Z}}$ is strictly stationary with $\mathrm{E}(\xi_{0,n})=0$, $\mathrm{E}(\xi_{0,n}^2)=1$ and $\sup_{n\geq 1}\mathrm{E}(|\xi_{0,n}|^\nu)<\infty$ for all $\nu\geq 1$, and is independent of the available sample $\boldsymbol{X}_1,\dots,\boldsymbol{X}_n$.

(M2) There exists a sequence $\ell_n\to\infty$ of strictly positive constants such that $\ell_n=o(n)$ and the sequence $(\xi_{i,n})_{i\in\mathbb{Z}}$ is ℓ_n-dependent, i.e., $\xi_{i,n}$ is independent of $\xi_{i+h,n}$ for all $h>\ell_n$ and $i\in\mathbb{N}$.

(M3) There exists a function $\varphi:\mathbb{R}\to[0,1]$, symmetric around 0, continuous at 0, satisfying $\varphi(0)=1$ and $\varphi(x)=0$ for all $|x|>1$ such that $\mathrm{E}(\xi_{0,n}\xi_{h,n})=\varphi(h/\ell_n)$ for all $h\in\mathbb{Z}$.

Let M be a large integer and let $(\xi_{i,n}^{(1)})_{i\in\mathbb{Z}},\dots,(\xi_{i,n}^{(M)})_{i\in\mathbb{Z}}$ be M independent copies of the same multiplier sequence. Then, following Bücher and Kojadinovic (2013) and Bücher et al. (2014), for any $m\in\{1,\dots,M\}$ and $(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d$, let

$$\hat{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u})=\frac{1}{\sqrt{n}}\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\xi_{i,n}^{(m)}\{\mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n}\leq\boldsymbol{u})-C_{1:n}(\boldsymbol{u})\},$$
$$\check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u})=\frac{1}{\sqrt{n}}\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}(\xi_{i,n}^{(m)}-\bar{\xi}^{(m)}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor})\mathbf{1}(\hat{\boldsymbol{U}}_i^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}\leq\boldsymbol{u}), \tag{3.1}$$

where $\bar{\xi}_{k:l}^{(m)}$ is the arithmetic mean of $\xi_{i,n}^{(m)}$ for $i\in\{k,\dots,l\}$.

The following proposition is a consequence of Theorem 1 in Holmes et al. (2013), Theorem 2.1 and the proof of Proposition 4.2 in Bücher and Kojadinovic (2013), as well as the proof of Proposition 4.3 in Bücher et al. (2014). It suggests interpreting the multiplier replicates $\hat{\mathbb{B}}_n^{(1)},\dots,\hat{\mathbb{B}}_n^{(M)}$ (resp. $\check{\mathbb{B}}_n^{(1)},\dots,\check{\mathbb{B}}_n^{(M)}$) as "almost" independent copies of $\mathbb{B}_n$ as n increases.

Proposition 3.1. *Assume that either*

(i) *the random vectors $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are i.i.d. with continuous margins and the sequences $(\xi_{i,n}^{(1)})_{i\in\mathbb{Z}}, \dots, (\xi_{i,n}^{(M)})_{i\in\mathbb{Z}}$ are independent copies of a multiplier sequence satisfying (M0),*

(ii) *or the random vectors $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are drawn from a strictly stationary sequence $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$ with continuous margins whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$ for some $a > 3 + 3d/2$, and $(\xi_{i,n}^{(1)})_{i\in\mathbb{Z}}, \dots, (\xi_{i,n}^{(M)})_{i\in\mathbb{Z}}$ are independent copies of a dependent multiplier sequence satisfying (M1)–(M3) with $\ell_n = O(n^{1/2-\varepsilon})$ for some $0 < \varepsilon < 1/2$.*

Then,

$$\left(\mathbb{B}_n, \hat{\mathbb{B}}_n^{(1)}, \dots, \hat{\mathbb{B}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{B}_C, \mathbb{B}_C^{(1)}, \dots, \mathbb{B}_C^{(M)}\right),$$
$$\left(\mathbb{B}_n, \check{\mathbb{B}}_n^{(1)}, \dots, \check{\mathbb{B}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{B}_C, \mathbb{B}_C^{(1)}, \dots, \mathbb{B}_C^{(M)}\right)$$

in $\{\ell^\infty(\Delta \times [0,1]^d)\}^{M+1}$, where $\mathbb{B}_C$ is given in (2.12) and $\mathbb{B}_C^{(1)}, \dots, \mathbb{B}_C^{(M)}$ are independent copies of $\mathbb{B}_C$.

Starting from the quantities defined above, we shall now define appropriate multiplier replicates under H_0 of the vector of processes $\mathbb{T}_n$ defined in (2.5). From (2.9), we see that to do so, we first need to define multiplier replicates of the processes $\mathbb{S}_{n,A}$, $A \subseteq D$, $|A| \geq 1$, defined in (2.8). From (2.13) and Proposition 3.1, natural candidates would be the processes $(s,t) \mapsto \psi_{C,A}\{\hat{\mathbb{B}}_n^{(m)}(s,t,\cdot)\}$ or the processes $(s,t) \mapsto \psi_{C,A}\{\check{\mathbb{B}}_n^{(m)}(s,t,\cdot)\}$, $m \in \{1,\dots,M\}$, where the map $\psi_{C,A}$ is defined in (2.14). These however still depend on the unknown copula C. The latter could be estimated either by $C_{1:n}$ or by $C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}$, which led us to consider the following two computable versions instead:

$$\hat{\mathbb{S}}_{n,A}^{(m)}(s,t) = \psi_{C_{1:n},A}\{\hat{\mathbb{B}}_n^{(m)}(s,t,\cdot)\}, \quad \check{\mathbb{S}}_{n,A}^{(m)}(s,t) = \psi_{C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor},A}\{\check{\mathbb{B}}_n^{(m)}(s,t,\cdot)\}, \qquad (s,t) \in \Delta.$$

The processes $\check{\mathbb{S}}_{n,A}^{(m)}$ were found to lead to better behaved tests than the $\hat{\mathbb{S}}_{n,A}^{(m)}$ in our Monte Carlo experiments, which is why, from now on, we focus solely on the former. It is easy to verify that the $\check{\mathbb{S}}_{n,A}^{(m)}$ can be rewritten as

$$\check{\mathbb{S}}_{n,A}^{(m)}(s,t) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (\xi_{i,n}^{(m)} - \bar{\xi}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{(m)}) \mathcal{I}_{C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor},A}(\hat{\boldsymbol{U}}_i^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}),$$

where, for any $\boldsymbol{u} \in [0,1]^d$,

$$\mathcal{I}_{C,A}(\boldsymbol{u}) = \psi_{C,A}\{\mathbf{1}(\boldsymbol{u} \leq \cdot)\} = \prod_{l\in A}(1-u_l) - \int_{[0,1]^d} \sum_{j\in A} \prod_{l\in A\setminus\{j\}} (1-v_l)\mathbf{1}(u_j \leq v_j)\mathrm{d}C(\boldsymbol{v}). \quad (3.2)$$

Next, by analogy with (2.9), for any $m \in \{1,\dots,M\}$, $A \subseteq D$, $|A| \geq 1$, let

$$\check{\mathbb{T}}_{n,A}^{(m)}(s) = \lambda_n(s,1)\check{\mathbb{S}}_{n,A}^{(m)}(0,s) - \lambda_n(0,s)\check{\mathbb{S}}_{n,A}^{(m)}(s,1), \qquad s \in [0,1],$$

and let $\check{\mathbb{T}}_n^{(m)}$ be the corresponding version of $\mathbb{T}_n$ in (2.5). Finally, for some continuous function $f : \mathbb{R}^{2^d-1} \to \mathbb{R}$, let $\check{S}_{n,f}^{(m)} = \sup_{s\in[0,1]} |f\{\check{\mathbb{T}}_n^{(m)}(s)\}|$ by analogy with (2.6). Interpreting the $\check{S}_{n,f}^{(m)}$ as multiplier replicates of $S_{n,f}$ under H_0, it is natural to compute an approximate p-value for the test as

$$\frac{1}{M}\sum_{m=1}^{M} \mathbf{1}\left(\check{S}_{n,f}^{(m)} \geq S_{n,f}\right). \tag{3.3}$$

The null hypothesis is rejected if the estimated p-value is smaller than the desired significance level.

The following result, proved in Appendix C, can be combined with Proposition F.1 in Bücher and Kojadinovic (2013) to show that a test based on $S_{n,f}$ whose p-value is computed as in (3.3) will hold its level asymptotically as $n \to \infty$ followed by $M \to \infty$.

Proposition 3.2. *Under the conditions of Proposition 3.1, for any $A \subseteq D$, $|A| \geq 1$,*

$$\left(\mathbb{S}_{n,A}, \check{\mathbb{S}}_{n,A}^{(1)}, \dots, \check{\mathbb{S}}_{n,A}^{(M)}\right) \rightsquigarrow \left(\mathbb{S}_{C,A}, \mathbb{S}_{C,A}^{(1)}, \dots, \mathbb{S}_{C,A}^{(M)}\right)$$

in $\{\ell^\infty(\Delta)\}^{M+1}$, where, for any $(s,t) \in \Delta$, $\mathbb{S}_{C,A}(s,t) = \psi_{C,A}\{\mathbb{B}_C(s,t,\cdot)\}$ and $\mathbb{S}_{C,A}^{(1)}, \dots, \mathbb{S}_{C,A}^{(M)}$ are independent copies of $S_{C,A}$. As a consequence,

$$\left(\mathbb{T}_n, \check{\mathbb{T}}_n^{(1)}, \dots, \check{\mathbb{T}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{T}_C, \mathbb{T}_C^{(1)}, \dots, \mathbb{T}_C^{(M)}\right)$$

in $\{\ell^\infty([0,1])\}^{(2^d-1)(M+1)}$, where $\mathbb{T}_C$ is given in (2.16) and $\mathbb{T}_C^{(1)}, \dots, \mathbb{T}_C^{(M)}$ are independent copies of T_C, and, for any continuous function $f : \mathbb{R}^{2^d-1} \to \mathbb{R}$,

$$\left(S_{n,f}, \check{S}_{n,f}^{(1)}, \dots, \check{S}_{n,f}^{(M)}\right) \rightsquigarrow \left(S_{C,f}, S_{C,f}^{(1)}, \dots, S_{C,f}^{(M)}\right)$$

in $\mathbb{R}^{M+1}$, where $S_{C,f} = \sup_{s\in[0,1]} |f\{\mathbb{T}_C(s)\}|$ and $S_{C,f}^{(1)}, \dots, S_{C,f}^{(M)}$ are independent copies of $S_{C,f}$.

The finite-sample behavior of the tests under consideration based on the processes $\check{\mathbb{S}}_{n,A}^{(m)}$ is not however completely satisfactory: the tests appear too liberal for multivariate time series with strong cross sectional dependence. This prompted us to try other asymptotically equivalent versions of the $\check{\mathbb{S}}_{n,A}^{(m)}$. Under an additional assumption on the partial derivatives of the copula, the generic test statistic $S_{n,f}$ defined in (2.6) can be written under H_0 as a functional of the *two-sided sequential empirical copula process* studied in Bücher and Kojadinovic (2013), and could therefore be bootstrapped via the multiplier processes defined in (4.4) of Bücher et al. (2014). Without imposing any condition on the partial derivatives of the copula, the latter remark, led us to consider, instead of the processes

$$\check{\mathbb{S}}_{n,A}^{(m)}(s,t) = \phi_A\{\check{\mathbb{B}}_n^{(m)}(s,t,\cdot)\} - \int_{[0,1]^d} \sum_{j\in A} \prod_{l\in A\setminus\{j\}} (1-v_l)\check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{v}^{\{j\}})\mathrm{d}C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{v}), \tag{3.4}$$

the processes

$$\tilde{\mathbb{S}}^{(m)}_{n,b_n,A}(s,t) = \phi_A\{\breve{\mathbb{B}}^{(m)}_n(s,t,\cdot)\} - \int_{[0,1]^d} \sum_{j\in A} \prod_{l\in A\setminus\{j\}} (1-v_l)\tilde{\mathbb{B}}^{(m)}_{n,b_n,j}(s,t,v_j)\mathrm{d}C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{v}), \tag{3.5}$$

where, for any $j\in D$, $\tilde{\mathbb{B}}^{(m)}_{n,b_n,j}$ is a linearly smoothed version of $(s,t,u)\mapsto \breve{\mathbb{B}}^{(m)}_n(s,t,\boldsymbol{u}_j)$ with $\boldsymbol{u}_j$ the vector of $[0,1]^d$ whose components are all equal to 1 except the jth one which is equal to u, and b_n a strictly positive sequence of constants converging to 0. Specifically,

$$\tilde{\mathbb{B}}^{(m)}_{n,b_n,j}(s,t,v) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (\xi^{(m)}_{i,n} - \bar{\xi}^{(m)}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor})\mathcal{L}_{b_n}(\hat{U}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}_{ij}, v), \qquad (s,t,v)\in\Delta\times[0,1],$$

where

$$\mathcal{L}_{b_n}(u,v) = \frac{u_+\wedge v - u_-\wedge v}{u_+ - u-}, \qquad u,v\in[0,1],$$

with $u_+ = (u+b_n)\wedge 1$ and $u_- = (u-b_n)\vee 0$. It is easy to verify that, for any $u\in[0,1]$, $\mathcal{L}_{b_n}(u,\cdot)$ differs from $\mathbf{1}(u\le\cdot)$ only on the interval (u_-,u_+) on which it linearly increases from 0 to 1.

Notice that (3.5) can be rewritten as

$$\tilde{\mathbb{S}}^{(m)}_{n,b_n,A}(s,t) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (\xi^{(m)}_{i,n} - \bar{\xi}^{(m)}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor})\mathcal{I}_{b_n,C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor},A}(\hat{\boldsymbol{U}}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}_i),$$

where, for any $\boldsymbol{u}\in[0,1]^d$,

$$\mathcal{I}_{b_n,C,A}(\boldsymbol{u}) = \prod_{l\in A}(1-u_l) - \int_{[0,1]^d} \sum_{j\in A} \prod_{l\in A\setminus\{j\}} (1-v_l)\mathcal{L}_{b_n}(u_j,v_j)\mathrm{d}C(\boldsymbol{v}). \tag{3.6}$$

For any $m\in\{1,\dots,M\}$, let $\tilde{\mathbb{T}}^{(m)}_{n,b_n}$ and $\tilde{S}^{(m)}_{n,b_n,f}$ be the analogues of $\breve{\mathbb{T}}^{(m)}_n$ and $\breve{S}^{(m)}_{n,f}$, respectively, defined from the processes $\tilde{\mathbb{S}}^{(m)}_{n,b_n,A}$ in (3.5). The following result, proved in Appendix C, is then the analogue of Proposition 3.2 above.

Proposition 3.3. *If $b_n = o(n^{-1/2})$, Proposition 3.2 holds with $\breve{\mathbb{S}}^{(m)}_{n,A}$ replaced by $\tilde{\mathbb{S}}^{(m)}_{n,b_n,A}$, $\breve{\mathbb{T}}^{(m)}_n$ replaced by $\tilde{\mathbb{T}}^{(m)}_{n,b_n}$ and $\breve{S}^{(m)}_{n,f}$ replaced by $\tilde{S}^{(m)}_{n,b_n,f}$.*

Finally, notice that it is possible to consider a version of the above construction in which the smoothing sequence is $b_{\lfloor nt\rfloor-\lfloor ns\rfloor}$ instead of b_n. We focused above only on the latter approach as it led to better behaved tests in our Monte Carlo experiments.

3.2 Estimating the asymptotic null distribution

When the function f used in the definition of $S_{n,f}$ in (2.6) is linear, Corollary 2.2 gives conditions under which, provided $\sigma^2_{C,f} = \mathrm{var}[f\circ\psi_C\{\mathbb{B}_C(0,1,\cdot)\}] > 0$, the weak limit of $\sigma^{-1}_{C,f}S_{n,f}$ under H_0 is equal in distribution to $\sup_{s\in[0,1]}|\mathbb{U}(s)|$. The distribution of the latter random variable can be approximated very well (this aspect is discussed in more detail in Section 4). To be able to estimate an asymptotic p-value for $S_{n,f}$, it thus remains to estimate the unknown variance $\sigma^2_{C,f}$.

Let E_ξ and var_ξ denote the expectation and variance, respectively, conditional on the data. By analogy with the classical way of proceeding when estimating variances using resampling procedures (see e.g. Künsch, 1989; Shao, 2010), in our context, a first natural estimator of the unknown variance under H_0 is of the form

$$\check\sigma^2_{n,C,f} = \mathrm{var}_\xi[f \circ \psi_C\{\check{\mathbb{B}}_n^{(m)}(0,1,\cdot)\}], \tag{3.7}$$

where $\check{\mathbb{B}}_n^{(m)}$ is defined in (3.1). To simplify the notation, we shall drop the superscript (m) in the rest of this section. The previous estimator is not computable as C is unknown, which is why we will eventually consider the estimator $\check\sigma^2_{n,C_{1:n},f}$ instead.

To obtain a more explicit expression of $\check\sigma^2_{n,C,f}$, first, let

$$\mathcal{I}_C(\boldsymbol{u}) = \big(\mathcal{I}_{C,\{1\}}(\boldsymbol{u}), \mathcal{I}_{C,\{2\}}(\boldsymbol{u}), \dots, \mathcal{I}_{C,D}(\boldsymbol{u})\big), \qquad \boldsymbol{u} \in [0,1]^d, \tag{3.8}$$

where $\mathcal{I}_{C,A}$, $A \subseteq D$, $|A| \geq 1$, is defined in (3.2). From the linearity of $f \circ \psi_C$, we then obtain that

$$\begin{aligned}\check\sigma^2_{n,C,f} &= \mathrm{var}_\xi\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^n (\xi_{i,n} - \bar\xi_{1:n}) f \circ \mathcal{I}_C(\hat{\boldsymbol{U}}_i^{1:n})\right\} \\ &= \mathrm{var}_\xi\left[\frac{1}{\sqrt{n}}\sum_{i=1}^n \xi_{i,n}\left\{f \circ \mathcal{I}_C(\hat{\boldsymbol{U}}_i^{1:n}) - \frac{1}{n}\sum_{j=1}^n f \circ \mathcal{I}_C(\hat{\boldsymbol{U}}_j^{1:n})\right\}\right].\end{aligned}$$

Using the fact that, from (3.2) and (3.8),

$$\frac{1}{n}\sum_{i=1}^n f \circ \mathcal{I}_C(\hat{\boldsymbol{U}}_i^{1:n}) = \frac{1}{n}\sum_{i=1}^n f \circ \psi_C\{\mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n} \leq \cdot)\} = f \circ \psi_C(C_{1:n}),$$

we obtain that

$$\check\sigma^2_{n,C,f} = \frac{1}{n}\sum_{i,j=1}^n \mathrm{E}_\xi(\xi_{i,n}\xi_{j,n}) f\left\{\mathcal{I}_C(\hat{\boldsymbol{U}}_i^{1:n}) - \psi_C(C_{1:n})\right\} f\left\{\mathcal{I}_C(\hat{\boldsymbol{U}}_j^{1:n}) - \psi_C(C_{1:n})\right\}.$$

On one hand, should the sequence $(\xi_{i,n})_{i\in\mathbb{Z}}$ be an i.i.d. multiplier sequence, that is, should it satisfy (M0), unsurprisingly, the above estimator simplifies to

$$\check\sigma^2_{n,C,f} = \frac{1}{n}\sum_{i=1}^n \left[f\left\{\mathcal{I}_C(\hat{\boldsymbol{U}}_i^{1:n}) - \psi_C(C_{1:n})\right\}\right]^2. \tag{3.9}$$

On the other hand, if the multiplier sequence satisfies (M1)–(M3), one obtains

$$\check\sigma^2_{n,C,f} = \frac{1}{n}\sum_{i,j=1}^n \varphi\left(\frac{i-j}{\ell_n}\right) f\left\{\mathcal{I}_C(\hat{\boldsymbol{U}}_i^{1:n}) - \psi_C(C_{1:n})\right\} f\left\{\mathcal{I}_C(\hat{\boldsymbol{U}}_j^{1:n}) - \psi_C(C_{1:n})\right\}, \tag{3.10}$$

which has the form of the HAC kernel estimator of de Jong and Davidson (2000).

Very naturally, once C has been replaced by $C_{1:n}$, we use the form in (3.9) (resp. (3.10)) for serially independent (resp. weakly dependent) observations. The following result, proved in Appendix D, establishes the consistency of $\check\sigma^2_{n,C_{1:n},f}$ under H_0.

Proposition 3.4. *Assume that $f : \mathbb{R}^{2^d-1} \to \mathbb{R}$ in the definition of* (2.6) *is linear and that either*

(i) *the random vectors $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are i.i.d. with continuous margins,*

(ii) *or the random vectors $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ are drawn from a strictly stationary sequence $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$ with continuous margins whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$ for some $a > 6$, and $\ell_n = O(n^{1/2-\varepsilon})$ for some $0 < \varepsilon < 1/2$ such that, additionally, φ defined in (M3) is twice continuously differentiable on $[-1, 1]$ with $\varphi''(0) \neq 0$ and is Lipschitz continuous on $\mathbb{R}$.*

Then, $\check\sigma^2_{n,C_{1:n},f} \xrightarrow{\mathrm{P}} \sigma^2_{C,f}$. As a consequence, the weak limit of $\check\sigma^{-1}_{n,C_{1:n},f} S_{n,f}$ is equal in distribution to $\sup_{s\in[0,1]} |\mathbb{U}(s)|$.

As in the previous subsection, better behaved tests are obtained if (3.6) is used instead of (3.2) in the above developments. Let

$$\mathcal{I}_{b_n,C}(\boldsymbol{u}) = \left(\mathcal{I}_{b_n,C,\{1\}}(\boldsymbol{u}), \mathcal{I}_{b_n,C,\{2\}}(\boldsymbol{u}), \dots, \mathcal{I}_{b_n,C,D}(\boldsymbol{u})\right), \qquad \boldsymbol{u} \in [0,1]^d,$$

and let $\tilde\sigma^2_{n,b_n,C_{1:n},f}$ be the corresponding estimator of $\sigma^2_{C,f}$. Proceeding as above, for serially independent data, the appropriate form of $\tilde\sigma^2_{n,b_n,C_{1:n},f}$ is

$$\tilde\sigma^2_{n,b_n,C_{1:n},f} = \frac{1}{n} \sum_{i=1}^{n} \left[f\left\{ \mathcal{I}_{b_n,C_{1:n}}(\hat{\boldsymbol{U}}_i^{1:n}) - \bar{\mathcal{I}}_{b_n,C_{1:n}} \right\} \right]^2, \tag{3.11}$$

where $\bar{\mathcal{I}}_{b_n,C_{1:n}} = n^{-1}\sum_{i=1}^n \mathcal{I}_{b_n,C_{1:n}}(\hat{\boldsymbol{U}}_i^{1:n})$, while, for weakly dependent observations,

$$\tilde\sigma^2_{n,b_n,C_{1:n},f} = \frac{1}{n} \sum_{i,j=1}^{n} \varphi\left(\frac{i-j}{\ell_n}\right) f\left\{ \mathcal{I}_{b_n,C_{1:n}}(\hat{\boldsymbol{U}}_i^{1:n}) - \bar{\mathcal{I}}_{b_n,C_{1:n}} \right\} f\left\{ \mathcal{I}_{b_n,C_{1:n}}(\hat{\boldsymbol{U}}_j^{1:n}) - \bar{\mathcal{I}}_{b_n,C_{1:n}} \right\}. \tag{3.12}$$

The following analogue of Proposition 3.4 is proved in Appendix D.

Proposition 3.5. *If $b_n = o(n^{-1/2})$, Proposition 3.4 holds with $\check\sigma^2_{n,C_{1:n},f}$ replaced with $\tilde\sigma^2_{n,b_n,C_{1:n},f}$.*

3.3 Estimation of the bandwidth parameter ℓ_n

When the available observations are weakly dependent, both the approach based on resampling presented in Section 3.1 and that based on the estimation of the asymptotic null distribution discussed in Section 3.2 require the choice of the bandwidth parameter ℓ_n. The latter quantity appears in the definition of the dependent multiplier sequences and, as mentioned in Bücher and Kojadinovic (2013), plays a role somehow analogous to that of the block length in the block bootstrap. The value of ℓ_n is therefore expected to have a crucial influence on the finite-sample performance of the two versions of the test based on $S_{n,f}$ described previously.

The aim of this subsection is to propose an estimator of ℓ_n in the spirit of that investigated in Paparoditis and Politis (2001) and Politis and White (2004), among others, for other resampling schemes. By analogy with (3.7), we start from the non computable estimator of $\sigma^2_{C,f}$ defined by

$$\sigma^2_{n,C,f} = \mathrm{var}_\xi[f \circ \psi_C\{\bar{\mathbb{B}}_n(0,1,\cdot)\}], \tag{3.13}$$

where

$$\bar{\mathbb{B}}_n(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \xi_{i,n}\{\mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}) - C(\boldsymbol{u})\}, \qquad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d,$$

and $(\xi_{i,n})_{i\in\mathbb{Z}}$ is a dependent multiplier sequence. Proceeding as for (3.7), it is easy to verify that

$$\sigma^2_{n,C,f} = \frac{1}{n}\sum_{i,j=1}^{n} \varphi\left(\frac{i-j}{\ell_n}\right) f\left\{\mathcal{I}_C(\boldsymbol{U}_i) - \psi_C(C)\right\} f\left\{\mathcal{I}_C(\boldsymbol{U}_j) - \psi_C(C)\right\}. \tag{3.14}$$

Under the conditions of Proposition 3.4 (ii) and from the fact that the random variables $|f \circ \mathcal{I}_C(\boldsymbol{U}_i)|$ are bounded by $\sup_{\boldsymbol{x}\in[-1,1]^{2^d-1}} |f(\boldsymbol{x})| < \infty$ (since $\sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C,A}(\boldsymbol{u})| \leq 1$ for all $A \subseteq D$ $|A| \geq 1$), we can for instance apply Lemmas 3.12 and 3.13 in Bühlmann (1993) (see also Proposition 2.1 in Shao (2010) and Propositions 5.1 and 5.2 in Bücher and Kojadinovic (2013)) to obtain that

$$\mathrm{E}(\sigma^2_{n,C,f}) - \sigma^2_{C,f} = \frac{\Gamma}{\ell_n^2} + o(\ell_n^{-2}) \qquad \text{and} \qquad \mathrm{var}(\sigma^2_{n,C,f}) = \frac{\ell_n}{n}\Delta + o(\ell_n/n),$$

where $\Gamma = \varphi''(0)/2\sum_{k=-\infty}^{\infty} k^2\tau(k)$ with $\tau(k) = \mathrm{cov}\{f \circ \mathcal{I}_C(\boldsymbol{U}_0), f \circ \mathcal{I}_C(\boldsymbol{U}_k)\}$, and $\Delta = 2\sigma^4_{C,f}\int_{-1}^{1}\varphi(x)^2\mathrm{d}x$. As a consequence, the mean squared error of $\sigma^2_{n,C,f}$ is

$$\mathrm{MSE}(\sigma^2_{n,C,f}) = \frac{\Gamma^2}{\ell_n^4} + \Delta\frac{\ell_n}{n} + o(\ell_n^{-4}) + o(\ell_n/n). \tag{3.15}$$

Differentiating the function $x \mapsto \Gamma^2/x^4 + \Delta x/n$ and equating the derivative to zero, we obtain that the value of ℓ_n that minimizes the mean square error of $\sigma^2_{n,C,f}$ is, asymptotically,

$$\ell_n^{opt} = \left(\frac{4\Gamma^2}{\Delta}\right)^{1/5} n^{1/5}.$$

To estimate ℓ_n^{opt}, it is necessary to estimate the infinite sum $\sum_{k\in\mathbb{Z}} k^2\tau(k)$ as well as $\sigma^2_{C,f} = \sum_{k\in\mathbb{Z}}\tau(k)$ through a *pilot* estimate. To do so, we adapt the approach described in Paparoditis and Politis (2001, page 1111) and Politis and White (2004, Section 3) to the current context. Let $\hat{\tau}_n(k)$ be the sample autocovariance at lag k computed from the sequence $f \circ \mathcal{I}_{b_n,C_{1:n}}(\hat{\boldsymbol{U}}_1^{1:n}), \dots, f \circ \mathcal{I}_{b_n,C_{1:n}}(\hat{\boldsymbol{U}}_n^{1:n})$. Then, we estimate Γ and Δ by

$$\hat{\Gamma}_n = \varphi''(0)/2\sum_{k=-L}^{L}\lambda(k/L)k^2\hat{\tau}_n(k) \quad \text{and} \quad \hat{\Delta}_n = 2\left\{\sum_{k=-L}^{L}\lambda(k/L)\hat{\tau}_n(k)\right\}^2\left\{\int_{-1}^{1}\varphi(x)^2\mathrm{d}x\right\},$$

respectively, where $\lambda(x) = [\{2(1 - |x|)\} \vee 0] \wedge 1$, $x \in \mathbb{R}$, is the "flat top" (trapezoidal) kernel of Politis and Romano (1995) and L is an integer estimated by adapting the procedure described in Politis and White (2004, Section 3.2). Let $\hat{\varrho}_n(k)$ be the sample autocorrelation at lag k estimated from $f \circ \mathcal{I}_{b_n,C_{1:n}}(\hat{\boldsymbol{U}}_1^{1:n}), \dots, f \circ \mathcal{I}_{b_n,C_{1:n}}(\hat{\boldsymbol{U}}_n^{1:n})$. The parameter L is then taken as the smallest integer k after which $\hat{\varrho}_n(k)$ appears negligible. The latter is determined automatically by means of the algorithm described in detail in Politis and White (2004, Section 3.2).

4 Monte Carlo experiments

In the previous section, two ways to compute approximate p-values for generic change-point tests based on (2.6) were studied under the null. These asymptotic results do not however guarantee that such tests will behave satisfactorily in finite-samples, which is why additional numerical simulations are needed. In our experiments, we restricted attention to the three statistics given in (2.3). For each statistic $S_{n,i}$, $i \in \{1, 2, 3\}$, an approximate p-value was computed using either the resampling approach based on the processes in (3.5), or the estimated asymptotic null distribution based on variance estimators of the form (3.11) or (3.12). To distinguish between these two situations, we shall talk about *the test* $\tilde{S}_{n,i}$ and *the test* $S^a_{n,i}$, respectively, in the rest of the paper.

The experiments were carried out in the R statistical system (R Development Core Team, 2014) using the `copula` package (Hofert et al., 2013). The sequence b_n involved in both classes of tests was taken equal to $n^{-0.51}$. The only (asymptotically negligible) difference with the theoretical developments presented in the previous sections is that the rescaled maximal ranks in (2.2) were computed by dividing the ranks by $l - k + 2$ instead of $l - k + 1$. The studied tests are implemented in the R package `npcp` (Kojadinovic, 2014).

Data generating procedure A simple autoregressive model of order one was used to generate d-dimensional samples of size n in our Monte Carlo experiments. Apart from d and n, the other inputs of the procedure are a real $t \in (0, 1)$ determining the location of the possible change-point, two d-dimensional copulas C_1 and C_2, and a real $\beta \in [0, 1)$ controlling the strength of the serial dependence. The procedure used to generate a d-dimensional sample $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ then consists of:

1. generating independent d-dimensional random vectors $\boldsymbol{U}_i$, $i \in \{-100, \dots, 0, \dots, n\}$ such that $\boldsymbol{U}_i$, $i \in \{-100, \dots, 0, \dots, \lfloor nt \rfloor\}$ are i.i.d. from copula C_1 and $\boldsymbol{U}_i$, $i \in \{\lfloor nt \rfloor + 1, \dots, n\}$ are i.i.d. from copula C_2,
2. computing $\boldsymbol{\epsilon}_i = (\Phi^{-1}(U_{i1}), \dots, \Phi^{-1}(U_{id}))$, where Φ is the c.d.f. of the standard normal distribution,
3. setting $\boldsymbol{X}_{-100} = \boldsymbol{\epsilon}_{-100}$ and, for any $j \in D$, computing recursively

$$X_{ij} = \beta X_{i-1,j} + \epsilon_{ij}, \qquad i = -99, \dots, 0, \dots, n. \tag{4.1}$$

If the copulas C_1 and C_2 are chosen equal, the above procedure generates samples under H_0 defined in (1.1). Three possible values were considered for the parameter β controlling the strength of the serial dependence: 0 (serial independence), 0.25 (mild serial dependence), 0.5 (strong serial dependence). Samples under $H_{0,m} \cap (\neg H_{0,c})$, where $H_{0,m}$ and $H_{0,c}$ are defined in (1.2) and (1.3), respectively, were obtained by taking $C_1 \neq C_2$ and $t \in \{0.1, 0.25, 0.5\}$. Notice that when $\beta = 0$, the latter are samples under $H_{0,m} \cap H_{1,c}$, where

$$H_{1,c} : \exists \text{ distinct } C_1 \text{ and } C_2, \text{ and } t \in (0,1) \text{ such that}$$
$$\boldsymbol{X}_1, \dots, \boldsymbol{X}_{\lfloor nt \rfloor} \text{ have copula } C_1 \text{ and } \boldsymbol{X}_{\lfloor nt \rfloor+1}, \dots, \boldsymbol{X}_n \text{ have copula } C_2.$$

This is not the case anymore when $\beta > 0$ as the change in cross-sectional dependence is then gradual by (4.1).

Other factors of the experiments Five copula families were considered (the Clayton, the Gumbel–Hougaard, the Normal, the Frank and the Student), the cross-sectional dimensional d was taken in $\{2, 4\}$, and the values 50, 100, 200, 400 and 500 were used for n. To estimate the power of the tests, 1000 samples were generated under each combination of factors and all the tests were carried out at the 5% significance level.

Computation of the test statistics and of the corresponding p-values The data generating procedure above generates multivariate time series whose component series do not contain ties with probability one. Consequently, as explained in Section 2.2, $S_{n,2}$ is merely $S_{n,1}$ computed from the sample $-\boldsymbol{X}_1, \dots, -\boldsymbol{X}_n$. Furthermore, if $d = 2$, it is easy to see that $S_{n,1} = S_{n,2} = S_{n,3}$. However, it can be verified that only the approximate p-values for the tests $\tilde{S}_{n,1}$ and $\tilde{S}_{n,3}$ (resp. $S^a_{n,1}$ and $S^a_{n,3}$) will be equal. Indeed, the multiplier replicates based on the processes in (3.5) (resp. the variance estimators of the form (3.11) or (3.12)) computed from $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ do not coincide in general with those computed from $-\boldsymbol{X}_1, \dots, -\boldsymbol{X}_n$, even in dimension two.

From Proposition 3.5, we see that, to compute an asymptotic p-value for the tests $S^a_{n,i}$, it is necessary to be able to compute the c.d.f. of the random variable $\sup_{s \in [0,1]} |\mathbb{U}(s)|$. The distribution of the latter random variable is known as the Kolmogorov distribution. As classically done in other contexts, we approach this distribution by that of the statistic of the classical Kolmogorov–Smirnov goodness-of-fit test for a simple hypothesis. Specifically, we use the function `pkolmogorov1x` given in the code of the R function `ks.test`.

[Table 1 about here.]

[Table 2 about here.]

Empirical levels and power of the tests based on i.i.d. multipliers / a variance estimator of the form (3.11) Table 1 gives the empirical levels of the tests when the observations are serially independent. For the sake of brevity, the results are reported only for two copula families. Overall, we find that the tests $\tilde{S}_{n,i}$ with multiplier sequences satisfying (M0) (here standard normal sequences) hold there level rather well both for $d = 2$ and $d = 4$, and all the considered degrees of cross-sectional dependence. This is not the case for the tests $S^a_{n,i}$ which frequently appear way too liberal when the cross-sectional dependence is high.

Table 2 partially reports the percentages of rejection of the i.i.d. multiplier tests for serially independent observations generated under $H_{0,m} \cap H_{1,c}$ resulting from a change of the copula parameter within a copula family. The columns CvM give the results of the i.i.d. multiplier test based on the maximally selected Cramér–von Mises statistic studied in Bücher et al. (2014, with multiplier replicates of the form (4.6)) and implemented in the R package `npcp`. Overall, we find that the tests $\tilde{S}_{n,i}$ are more powerful than that studied in Bücher et al. (2014) for such scenarios, especially when the change in the copula occurs early or late. Among the tests $\tilde{S}_{n,i}$, we observed that the test $\tilde{S}_{n,3}$ (which coincides with the test $\tilde{S}_{n,1}$ in dimension two) led frequently to slightly higher rejection rates, although this conclusion is based on a limited number of simulation scenarios. The rejection rates of the tests $S^a_{n,i}$ with a variance estimator of the form (3.11) are not reported for the sake of brevity. They were found to be slightly less powerful than the tests $\tilde{S}_{n,i}$ when $\tau = 0.4$. For $\tau = 0.6$, a comparison of the two classes of tests is not necessarily meaningful as the tests $S^a_{n,i}$ were often found to be way too liberal under strong cross-sectional dependence.

[Table 3 about here.]

[Table 4 about here.]

[Table 5 about here.]

Empirical levels and power of the tests based on dependent multipliers / a variance estimator of the form (3.12) Part of Table 3 reports the empirical levels of the test $\tilde{S}_{n,1}$ when dependent multiplier sequences satisfying (M1)–(M3) are used. These sequences were generated using the "moving average approach" proposed initially in Bühlmann (1993, Section 6.2) and revisited in Bücher and Kojadinovic (2013, Section 6.1). A standard normal sequence was used for the required initial i.i.d. sequence. The kernel function κ in that approach was chosen to be the Parzen kernel defined by $\kappa_P(x) = (1-6x^2+6|x|^3)\mathbf{1}(|x| \leq 1/2)+2(1-|x|)^3\mathbf{1}(1/2 < |x| \leq 1)$, $x \in \mathbb{R}$, which amounts to choosing the function φ in (M3) as $x \mapsto (\kappa_P \star \kappa_P)(2x)/(\kappa_P \star \kappa_P)(0)$, where '$\star$' denotes the convolution operator. The value of the bandwidth parameter ℓ_n defined in (M2) was estimated using the data-driven procedure described in Section 3.3. The same value of ℓ_n was used to carry out the test $S^a_{n,1}$ relying on a variance estimator of the form (3.12).

From the first three vertical blocks of Table 3, we see that an increase in the degree of serial dependence (controlled by β) appears to result in a small inflation of the empirical levels of the test $\tilde{S}_{n,1}$. As expected, the situation improves as n increases from 100 to 400. As previously, the test $S^a_{n,1}$ is way too liberal when the cross-sectional dependence is high.

The last vertical block of Table 3 reports, for strongly serially dependent observations, the empirical levels of the test $\tilde{S}_{n,1}$ based on i.i.d. multipliers, as well as those of the test $S^a_{n,1}$ based on an inappropriate variance estimator of the form (3.11). As expected, both tests strongly fail to hold their level.

Table 4 partially reports the rejection percentages of the tests based on dependent multipliers / a variance estimator of the form (3.12) for observations generated under $H_{0,m} \cap (\neg H_{0,c})$ resulting from a change of the copula parameter within a copula family.

The rejection rates of the test $S^a_{n,1}$ should be considered with care when $\tau = 0.6$ as that test was found to be way too liberal under strong cross-sectional dependence. Despite that issue, the test $\tilde{S}_{n,1}$ appears almost always more powerful than the test $S^a_{n,1}$. Also, as it could have been expected, the presence of serial dependence ($\beta = 0.5$) leads to lower rejection percentages when compared with serial independence ($\beta = 0$). Finally, comparing the results for the test $\tilde{S}_{n,1}$ when $\beta = 0$ with the analogue results reported in Table 2 reveals that, rather naturally, the use of dependent multipliers in the case of serially independent observations results in a small loss of power.

We end this section by a comparison of the tests $\tilde{S}_{n,1}$ and $S^a_{n,1}$ with the similar test studied in Wied et al. (2014). To do so, we reproduced one of the experiments carried out in the latter reference. The results are reported in Table 5 and confirm that tests for change-point detection based on (2.1) are potentially substantially more powerful than tests based on (2.7).

5 Practical recommendations and illustration

Based on the previously reported numerical experiments, we recommend, among the tests $\tilde{S}_{n,i}$ and $S^a_{n,i}$, the tests $\tilde{S}_{n,i}$. Indeed, the tests $S^a_{n,i}$ did not hold their level well in the case of strong cross-sectional dependence. Furthermore, because of their form, the tests $S^a_{n,i}$ might suffer from some of the practical issues described in Shao and Zhang (2010), and, in future research, it might be of interest to study a *self-normalization* version of these as advocated in the latter reference.

The pros and cons of the tests $\tilde{S}_{n,i}$ compared with the test studied in Bücher et al. (2014) are as follows. The tests $\tilde{S}_{n,i}$ seem more powerful for alternatives involving a change in Spearman's rho at constant margins; they are also substantially faster to compute. Their main weakness is that, by construction, they have no power against alternatives involving a change in the copula at a constant value of Spearman's rho and constant margins.

Among the tests $\tilde{S}_{n,i}$, we recommend the test $\tilde{S}_{n,3}$, merely because of its slightly better finite-sample behavior in our simulations.

We end this work by a brief illustration of the studied tests on real financial observations. Specifically, we consider a trivariate version of the data analyzed in Dehling et al. (2014, Section 7). The observations consist of $n = 990$ daily logreturns computed from the DAX, the CAC 40 and the Standard and Poor 500 indices for the years 2006–2009. An approximate p-value of 0.045 was obtained for the test $\tilde{S}_{n,3}$ with dependent multipliers, providing some evidence against H_0. It is however important to bear in mind that it is only under the assumption that $H_{0,m}$ in (1.2) holds that it would be fully justified to decide to reject $H_{0,c}$ in (1.3).

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A Proof of Proposition 2.1

Let us first introduce some additional notation. For integers $1 \leq k \leq l \leq n$, let $H_{k:l}$ denote the empirical c.d.f. of the unobservable sample $\boldsymbol{U}_k, \dots, \boldsymbol{U}_l$ and let $H_{k:l,1}, \dots, H_{k:l,d}$ denote its margins. The corresponding empirical quantile functions are

$$H_{k:l,j}^{-1}(u) = \inf\{v \in [0,1] : H_{k:l,j}(v) \geq u\}, \qquad u \in [0,1], j \in D.$$

Finally, for any $\boldsymbol{u} \in [0,1]^d$, let

$$\boldsymbol{h}_{k:l}(\boldsymbol{u}) = \big(H_{k:l,1}(u_1), \dots, H_{k:l,d}(u_d)\big) \tag{A.1}$$

and

$$\boldsymbol{h}_{k:l}^{-1}(\boldsymbol{u}) = \big(H_{k:l,1}^{-1}(u_1), \dots, H_{k:l,d}^{-1}(u_d)\big). \tag{A.2}$$

By convention, all the quantities defined above are taken equal to zero if $k > l$.

Proof of Proposition 2.1. Fix $A \subseteq D$, $|A| \geq 1$, and $(s,t) \in \Delta$ such that $\lfloor ns \rfloor < \lfloor nt \rfloor$. On one hand, from (2.8) and by linearity of ϕ_A defined in (2.4), we have

$$\mathbb{S}_{n,A}(s,t) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \prod_{j \in A} \{1 - H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor,j}(U_{ij})\} - \sqrt{n}\lambda_n(s,t)\phi_A(C),$$

where we have used the fact that $\hat{U}_{ij}^{\lfloor ns \rfloor+1:\lfloor nt \rfloor} = H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor,j}(U_{ij})$ for all $j \in D$ and all $i \in \{\lfloor ns \rfloor+1, \dots, \lfloor nt \rfloor\}$. On the other hand,

$$\psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\} = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \prod_{j \in A} (1 - U_{ij}) - \sqrt{n}\lambda_n(s,t)\phi_A(C) \\ - \int_{[0,1]^d} \sum_{j \in A} \prod_{l \in A \setminus \{j\}} (1 - v_l) \mathbb{B}_n(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}C(\boldsymbol{v}).$$

Next, let $\pi(\boldsymbol{u}) = \prod_{j \in A}(1 - u_j)$, $\boldsymbol{u} \in \mathbb{R}^d$. Then, fix $\boldsymbol{u} \in [0,1]^d$, and, for any $x \in [0,1]$, let $\boldsymbol{w}_{\boldsymbol{u}}(x) = \boldsymbol{u} + x\{\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u}) - \boldsymbol{u}\}$ and let $g(x) = \pi\{\boldsymbol{w}_{\boldsymbol{u}}(x)\}$, where $\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}$ is defined in (A.1). The function g is clearly continuously differentiable on $[0,1]$. By the mean value theorem, there exists $x_{\boldsymbol{u},n,s,t}^* \in (0,1)$ such that $g(1) - g(0) = g'(x_{\boldsymbol{u},n,s,t}^*)$, that is, such that

$$\pi\{\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u})\} - \pi(\boldsymbol{u}) = \sum_{j \in A} \dot{\pi}_j[\boldsymbol{u} + x_{\boldsymbol{u},n,s,t}^*\{\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u}) - \boldsymbol{u}\}]\{H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor,j}(u_j) - u_j\}.$$

It follows that

$$\begin{aligned}\mathbb{S}_{n,A}(s,t) - \psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\} \\ = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \sum_{j \in A} \dot{\pi}_j[\boldsymbol{U}_i + x^*_{\boldsymbol{U}_i,n,s,t}\{\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{U}_i) - \boldsymbol{U}_i\}]\{H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor,j}(U_{ij}) - U_{ij}\} \\ - \int_{[0,1]^d} \sum_{j \in A} \dot{\pi}_j(\boldsymbol{v}) \mathbb{B}_n(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}C(\boldsymbol{v}).\end{aligned}$$

Notice that, by the triangle inequality and the fact that $\sup_{\boldsymbol{u} \in [0,1]^d} |\dot{\pi}_j(\boldsymbol{u})| \leq 1$, $j \in D$,

$$\sup_{(s,t) \in \Delta} |\mathbb{S}_{n,A}(s,t) - \psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\}| \leq 2|A| \sup_{(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d} |\mathbb{B}_n(s,t,\boldsymbol{u})|.$$

Next, fix $\varepsilon, \eta > 0$. Using the previous inequality and the fact that $\mathbb{B}_n$ vanishes when $s = t$ and is asymptotically uniformly equicontinuous in probability as a consequence of Lemma 2 in Bücher (2013), there exists $\delta \in (0,1)$ such that, for all sufficiently large n,

$$\begin{aligned}\mathrm{P}\left(\sup_{\substack{(s,t) \in \Delta \\ t-s<\delta}} |\mathbb{S}_{n,A}(s,t) - \psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\}| > \varepsilon \right) \\ \leq \mathrm{P}\left(2|A| \sup_{\substack{(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d \\ t-s<\delta}} |\mathbb{B}_n(s,t,\boldsymbol{u})| > \varepsilon \right) < \eta/2.\end{aligned}$$

To show (2.13), it remains therefore to prove that, for all sufficiently large n,

$$\mathrm{P}\left(\sup_{\substack{(s,t) \in \Delta \\ t-s\geq\delta}} |\mathbb{S}_{n,A}(s,t) - \psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\}| > \varepsilon \right) < \eta/2.$$

To show the above, we shall now prove that $\sup_{(s,t) \in \Delta^\delta} |\mathbb{S}_{n,A}(s,t) - \psi_{C,A}\{\mathbb{B}_n(s,t,\cdot)\}|$ converges in probability to zero, where $\Delta^\delta = \{(s,t) \in \Delta : t - s \geq \delta\}$. The latter supremum is smaller than $\sum_{j \in A}(I_{n,j} + II_{n,j})$, where

$$\begin{aligned}I_{n,j} \leq \sup_{(s,t) \in \Delta^\delta} \Bigg| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \big(\dot{\pi}_j[\boldsymbol{U}_i + x^*_{\boldsymbol{U}_i,n,s,t}\{\boldsymbol{h}_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{U}_i) - \boldsymbol{U}_i\}] - \dot{\pi}_j(\boldsymbol{U}_i) \big) \\ \times \{H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor,j}(U_{ij}) - U_{ij}\} \Bigg|\end{aligned}$$

and

$$II_{n,j} \leq \sup_{(s,t) \in \Delta^\delta} \Big| \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v}) \mathbb{B}_n(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{v}) - \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v}) \mathbb{B}_n(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}C(\boldsymbol{v}) \Big|.$$

Next, notice that

$$\begin{aligned}\sup_{(s,t,\boldsymbol{u}) \in \Delta^\delta \times [0,1]^d} |H_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\boldsymbol{u}) - C(\boldsymbol{u})| \\ \leq \sup_{(s,t,\boldsymbol{u}) \in \Delta^\delta \times [0,1]^d} |\mathbb{B}_n(s,t,\boldsymbol{u})| \times n^{-1/2} \times \sup_{(s,t) \in \Delta^\delta} \{\lambda_n(s,t)\}^{-1} \xrightarrow{\mathrm{P}} 0. \quad \text{(A.3)}\end{aligned}$$

Fix $j \in A$. Since the function $\dot{\pi}_j$ is continuous on $[0,1]^d$, by the continuous mapping theorem, $\sup_{(s,t,\boldsymbol{u})\in\Delta^\delta\times[0,1]^d} |\dot{\pi}_j[\boldsymbol{u} + x^*_{\boldsymbol{u},n,s,t}\{\boldsymbol{h}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - \boldsymbol{u}\}] - \dot{\pi}_j(\boldsymbol{u})| \xrightarrow{\mathrm{P}} 0$. Hence,

$$
\begin{aligned}
I_{n,j} \leq & \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} |\mathbb{B}_n(s,t,\boldsymbol{u})| \\
& \qquad \times \sup_{(s,t,\boldsymbol{u})\in\Delta^\delta\in[0,1]^d} |\dot{\pi}_j[\boldsymbol{u} + x^*_{\boldsymbol{u},n,s,t}\{\boldsymbol{h}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - \boldsymbol{u}\}] - \dot{\pi}_j(\boldsymbol{u})| \xrightarrow{\mathrm{P}} 0.
\end{aligned}
$$

It thus remains to show that $II_{n,j} \xrightarrow{\mathrm{P}} 0$. The latter is mostly a consequence of Lemma A.1 below. First, notice that (A.3) implies that $H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \xrightarrow{\mathrm{P}} C$ in $\ell^\infty(\Delta^\delta\times[0,1]^d)$. Hence, $(\mathbb{B}_n, H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}) \rightsquigarrow (\mathbb{B}_C, C)$ in $\ell^\infty(\Delta^\delta\times[0,1]^d)$. Next, combining the previous weak convergence with Lemma 3 in Holmes et al. (2013) and the continuous mapping theorem, we obtain that the finite-dimensional distributions of $(\mathbb{A}_{n,j}, \mathbb{B}_n)$ converge weakly to those of $(\mathbb{A}_{C,j}, \mathbb{B}_C)$, where $\mathbb{A}_{n,j}$ and $\mathbb{A}_{C,j}$ are defined in Lemma A.1. The fact that $(\mathbb{A}_{n,j}, \mathbb{B}_n) \rightsquigarrow (\mathbb{A}_{C,j}, \mathbb{B}_C)$ in $\{\ell^\infty(\Delta^\delta\times[0,1]^d)\}^2$ then follows from Lemma A.1 below and the fact that marginal asymptotic tightness implies joint asymptotic tightness. The latter weak convergence combined with the continuous mapping theorem finally implies that $II_{n,j} \xrightarrow{\mathrm{P}} 0$, which completes the proof. ■

Lemma A.1. *For any $j \in D$ and $\delta \in (0,1)$, $\mathbb{A}_{n,j} \rightsquigarrow \mathbb{A}_{C,j}$ in $\ell^\infty(\Delta^\delta)$, where*

$$
\begin{aligned}
\mathbb{A}_{n,j}(s,t) &= \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v})\mathbb{B}_n(s,t,\boldsymbol{v}^{\{j\}})\mathrm{d}H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{v}), \\
\mathbb{A}_{C,j}(s,t) &= \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v})\mathbb{B}_C(s,t,\boldsymbol{v}^{\{j\}})\mathrm{d}C(\boldsymbol{v}).
\end{aligned}
\tag{A.4}
$$

Proof. Fix $j \in D$ and $\delta \in (0,1)$. To prove the desired result, we shall show that conditions (i) and (ii) of Theorem 2.1 in Kosorok (2008) hold. First, recall that from (A.3), $H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \xrightarrow{\mathrm{P}} C$ in $\ell^\infty(\Delta^\delta\times[0,1]^d)$. Then, from the fact that $\mathbb{B}_n \rightsquigarrow \mathbb{B}_C$ in $\ell^\infty(\Delta\times[0,1]^d)$, we obtain that, for any $(s_1,t_1),\dots,(s_k,t_k) \in \Delta^\delta$,

$$
\begin{aligned}
&\big(\mathbb{B}_n(s_1,t_1,\cdot), H_{\lfloor ns_1\rfloor+1:\lfloor nt_1\rfloor}, \dots, \mathbb{B}_n(s_k,t_k,\cdot), H_{\lfloor ns_k\rfloor+1:\lfloor nt_k\rfloor}\big) \\
&\qquad\qquad\qquad \rightsquigarrow \big(\mathbb{B}_C(s_1,t_1,\cdot), C, \dots, \mathbb{B}_C(s_k,t_k,\cdot), C\big)
\end{aligned}
$$

in $\{\ell^\infty([0,1]^d)\}^{2k}$. From Lemma 3 in Holmes et al. (2013) and the continuous mapping theorem, the above implies that $\big(\mathbb{A}_{n,j}(s_1,t_1),\dots,\mathbb{A}_{n,j}(s_k,t_k)\big) \rightsquigarrow \big(\mathbb{A}_{C,j}(s_1,t_1),\dots,\mathbb{A}_{C,j}(s_k,t_k)\big)$ in $\mathbb{R}^k$. Hence, we have convergence of the finite-dimensional distributions, that is, condition (i) of Theorem 2.1 in Kosorok (2008) holds.

It remains to prove condition (ii) of Theorem 2.1 in Kosorok (2008). Specifically, we shall now show that $\mathbb{A}_{n,j}$ is $\|\cdot\|_1$-asymptotically uniformly equicontinuous in probability, which will complete the proof since Δ^δ is totally bounded by $\|\cdot\|_1$. By Problem 2.1.5 in van der Vaart and Wellner (2000), we need to show that, for any positive sequence $a_n \downarrow 0$,

$$
\sup_{\substack{(s,t),(s',t')\in\Delta^\delta \\ |s-s'|+|t-t'|\leq a_n}} |\mathbb{A}_{n,j}(s,t) - \mathbb{A}_{n,j}(s',t')| \xrightarrow{\mathrm{P}} 0. \tag{A.5}
$$

We bound the supremum on the left of the previous display by $I_n + II_n$, where

$$I_n = \sup_{\substack{(s,t),(s',t')\in\Delta^\delta \\ |s-s'|+|t-t'|\le a_n}} \left| \int_{[0,1]^d} \dot\pi_j(\boldsymbol{v})\mathbb{B}_n(s,t,\boldsymbol{v}^{\{j\}})\mathrm{d}H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{v}) - \int_{[0,1]^d} \dot\pi_j(\boldsymbol{v})\mathbb{B}_n(s',t',\boldsymbol{v}^{\{j\}})\mathrm{d}H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{v}) \right|$$

and

$$II_n = \sup_{\substack{(s,t),(s',t')\in\Delta^\delta \\ |s-s'|+|t-t'|\le a_n}} \left| \int_{[0,1]^d} \dot\pi_j(\boldsymbol{v})\mathbb{B}_n(s',t',\boldsymbol{v}^{\{j\}})\mathrm{d}H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{v}) - \int_{[0,1]^d} \dot\pi_j(\boldsymbol{v})\mathbb{B}_n(s',t',\boldsymbol{v}^{\{j\}})\mathrm{d}H_{\lfloor ns'\rfloor+1:\lfloor nt'\rfloor}(\boldsymbol{v}) \right|.$$

Now,

$$I_n \le \sup_{\boldsymbol{u}\in[0,1]^d} |\dot\pi_j(\boldsymbol{u})| \times \sup_{\substack{(s,t),(s',t')\in\Delta^\delta,\boldsymbol{u}\in[0,1]^d \\ |s-s'|+|t-t'|\le a_n}} |\mathbb{B}_n(s,t,\boldsymbol{u}) - \mathbb{B}_n(s',t',\boldsymbol{u})| \xrightarrow{\mathrm{P}} 0,$$

since $\mathbb{B}_n$ is asymptotically uniformly equicontinuous in probability as a consequence of Lemma 2 in Bücher (2013). Furthermore, II_n is smaller than

$$\sup_{\substack{(s,t),(s',t')\in\Delta^\delta \\ |s-s'|+|t-t'|\le a_n}} \left| \frac{1}{\lfloor nt\rfloor - \lfloor ns\rfloor} \left\{ \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \dot\pi_j(\boldsymbol{U}_i)\mathbb{B}_n(s',t',\boldsymbol{U}_i^{\{j\}}) - \sum_{i=\lfloor ns'\rfloor+1}^{\lfloor nt'\rfloor} \dot\pi_j(\boldsymbol{U}_i)\mathbb{B}_n(s',t',\boldsymbol{U}_i^{\{j\}}) \right\} \right|$$
$$+ \sup_{\substack{(s,t),(s',t')\in\Delta^\delta \\ |s-s'|+|t-t'|\le a_n}} \left| \left(\frac{1}{\lfloor nt\rfloor - \lfloor ns\rfloor} - \frac{1}{\lfloor nt'\rfloor - \lfloor ns'\rfloor} \right) \sum_{i=\lfloor ns'\rfloor+1}^{\lfloor nt'\rfloor} \dot\pi_j(\boldsymbol{U}_i)\mathbb{B}_n(s',t',\boldsymbol{U}_i^{\{j\}}) \right|,$$

which is smaller than

$$2 \times \sup_{\substack{(s,t),(s',t')\in\Delta^\delta \\ |s-s'|+|t-t'|\le a_n}} \frac{|\lfloor nt\rfloor - \lfloor nt'\rfloor| + |\lfloor ns\rfloor - \lfloor ns'\rfloor|}{\lfloor nt\rfloor - \lfloor ns\rfloor} \times \sup_{\boldsymbol{u}\in[0,1]^d} |\dot\pi_j(\boldsymbol{u})| \times \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} |\mathbb{B}_n(s,t,\boldsymbol{u})| \xrightarrow{\mathrm{P}} 0.$$

Hence, $II_n \xrightarrow{\mathrm{P}} 0$ and thus (A.5) holds, which completes the proof. ■

B Proof of Corollary 2.2

Proof. Starting from (2.9), using Proposition 2.1, the linearity of $\psi_{C,A}$ and (2.10), we obtain that, for any $A \subseteq D$, $|A| \ge 1$,

$$\sup_{s\in[0,1]} |\mathbb{T}_{n,A}(s) - \psi_{C,A}\{\mathbb{B}_n(0,s,\cdot) - \lambda(0,s)\mathbb{B}_n(0,1,\cdot)\}| = o_{\mathrm{P}}(1).$$

Hence, $\mathbb{T}_n$ has the same weak limit as $s \mapsto \psi_C\{\mathbb{B}_n(0,s,\cdot) - \lambda(0,s)\mathbb{B}_n(0,1,\cdot)\}$ and (2.15) follows from the continuous mapping theorem.

The second to last claim is a consequence of the continuous mapping theorem. To prove the last claim, it suffices to show that the Gaussian process $\sigma_{C,f}^{-1} f\{\mathbb{T}_C(\cdot)\}$ has the same covariance function as $\mathbb{U}$. For any, $s,t \in [0,1]$, we have

$$\begin{aligned}&\operatorname{cov}[\sigma_{C,f}^{-1} f\{\mathbb{T}_C(s)\}, \sigma_{C,f}^{-1} f\{\mathbb{T}_C(t)\}] \\ &\quad = \sigma_{C,f}^{-2} \operatorname{E}[f \circ \psi_C\{\mathbb{B}_C(0,s,\cdot) - s\mathbb{B}_C(0,1,\cdot)\} f \circ \psi_C\{\mathbb{B}_C(0,t,\cdot) - t\mathbb{B}_C(0,1,\cdot)\}]. \quad \text{(B.1)}\end{aligned}$$

By linearity of $f \circ \psi_C$ and Fubini's theorem, the expectation in the last display is equal to

$$f \circ \psi_C \left\{ \boldsymbol{u} \mapsto f \circ \psi_C \left(\boldsymbol{v} \mapsto \operatorname{E}[\{\mathbb{B}_C(0,s,\boldsymbol{u}) - s\mathbb{B}_C(0,1,\boldsymbol{u})\}\{\mathbb{B}_C(0,t,\boldsymbol{v}) - t\mathbb{B}_C(0,1,\boldsymbol{v})\}] \right) \right\},$$

that is,

$$(s \wedge t - st) f \circ \psi_C \left[\boldsymbol{u} \mapsto f \circ \psi_C \left\{ \boldsymbol{v} \mapsto \kappa_C(\boldsymbol{u},\boldsymbol{v}) \right\} \right] = (s \wedge t - st) \operatorname{var}[f \circ \psi_C\{\mathbb{B}_C(0,1,\cdot)\}],$$

where κ_C is defined in (2.11). Combining the previous display with (B.1), we obtain that $\operatorname{cov}[\sigma_{C,f}^{-1} f\{\mathbb{T}_C(s)\}, \sigma_{C,f}^{-1} f\{\mathbb{T}_C(t)\}] = (s \wedge t - st)$, which completes the proof. ■

C Proofs of Propositions 3.2 and 3.3

Proof of Proposition 3.2. We only show the first claim as the subsequent claims then mostly follow from the continuous mapping theorem. Also, we only provide the proof under (ii) in the statement of Proposition 3.1, the proof being simpler under (i). Fix $A \subseteq D$, $|A| \geq 1$. For any $(s,t) \in \Delta$, let $\mathbb{S}_{n,A}^{(m)}(s,t) = \psi_{C,A}\{\check{\mathbb{B}}_n^{(m)}(s,t,\cdot)\}$. Using the linearity of the map $\psi_{C,A}$ defined in (2.14), Proposition 3.1 and the continuous mapping theorem, we obtain that

$$\left(\mathbb{S}_{n,A}, \mathbb{S}_{n,A}^{(1)}, \dots, \mathbb{S}_{n,A}^{(M)}\right) \leadsto \left(\mathbb{S}_{C,A}, \mathbb{S}_{C,A}^{(1)}, \dots, \mathbb{S}_{C,A}^{(M)}\right)$$

in $\{\ell^\infty(\Delta)\}^{M+1}$. The first claim is thus proved if we show that, for any $m \in \{1,\dots,M\}$, $\sup_{(s,t)\in\Delta} |\check{\mathbb{S}}_{n,A}^{(m)}(s,t) - \mathbb{S}_{n,A}^{(m)}(s,t)|$ is $o_{\mathrm{P}}(1)$. Fix $m \in \{1,\dots,M\}$ and notice that the latter supremum is smaller than $2|A| \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} |\check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u})|$. We can therefore proceed analogously to the proof of Proposition 2.1. Fix $\varepsilon, \eta > 0$. Using the previous inequality as well as the fact that $\check{\mathbb{B}}_n^{(m)}$ is zero when $s = t$ and is asymptotically uniformly equicontinuous in probability as a consequence of Lemma A.3 in Bücher and Kojadinovic (2013), there exists $\delta \in (0,1)$ such that, for all sufficiently large n,

$$\mathrm{P}\left(\sup_{\substack{(s,t)\in\Delta \\ t-s<\delta}} |\check{\mathbb{S}}_{n,A}^{(m)}(s,t) - \mathbb{S}_{n,A}^{(m)}(s,t)| > \varepsilon \right) < \eta/2.$$

It remains therefore to prove that $\sup_{(s,t)\in\Delta^\delta} |\check{\mathbb{S}}_{n,A}^{(m)}(s,t) - \mathbb{S}_{n,A}^{(m)}(s,t)| \overset{\mathrm{P}}{\to} 0$, where $\Delta^\delta = \{(s,t) \in \Delta : t - s \geq \delta\}$. The latter supremum is smaller than

$$\sum_{j\in A} \sup_{(s,t)\in\Delta^\delta} \left| \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v}) \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{v}) - \int_{[0,1]^d} \dot{\pi}_j(\boldsymbol{v}) \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{v}^{\{j\}}) \mathrm{d}C(\boldsymbol{v}) \right|,$$

where $\dot\pi_j$ is the jth first order partial derivative of the function $\pi(\boldsymbol{u}) = \prod_{j\in A}(1-u_j)$, $\boldsymbol{u}\in\mathbb{R}^d$, introduced in the proof of Proposition 2.1. Fix $j\in A$. The jth summand in the previous display is smaller than $I_n + II_n$, where

$$I_n = \sup_{(s,t)\in\Delta^\delta}\Big|\int_{[0,1]^d}\dot\pi_j(\boldsymbol{v})\breve{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{v}^{\{j\}})\mathrm{d}C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{v}) - \breve{\mathbb{A}}_{n,j}^{(m)}(s,t)\Big|,$$
$$II_n = \sup_{(s,t)\in\Delta^\delta}\Big|\breve{\mathbb{A}}_{n,j}^{(m)}(s,t) - \int_{[0,1]^d}\dot\pi_j(\boldsymbol{v})\breve{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{v}^{\{j\}})\mathrm{d}C(\boldsymbol{v})\Big|,$$

and $\breve{\mathbb{A}}_{n,j}^{(m)}$ is defined analogously to the process $\mathbb{A}_{n,j}$ in (A.4) with $\mathbb{B}_n$ replaced by $\breve{\mathbb{B}}_n^{(m)}$. In addition, it can be verified that Lemma A.1 remains true if $\mathbb{B}_n$ and $\mathbb{B}_C$ are replaced by $\breve{\mathbb{B}}_n^{(m)}$ and $\mathbb{B}_C^{(m)}$, respectively, in its statement. It follows that we can proceed as at the end of proof of Proposition 2.1 to show that II_n above converges to zero in probability.

To show that $I_n \xrightarrow{\mathrm{P}} 0$, we use the fact that $I_n \le I_n' + I_n''$, where

$$I_n' = \sup_{(s,t)\in\Delta^\delta}\Big|\frac{1}{\lfloor nt\rfloor - \lfloor ns\rfloor}\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\big[\dot\pi_j\{\boldsymbol{h}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{U}_i)\} - \dot\pi_j(\boldsymbol{U}_i)\big]\,\breve{\mathbb{B}}_n^{(m)}\{s,t,\boldsymbol{h}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{U}_i)^{\{j\}}\}\Big|,$$
$$I_n'' = \sup_{(s,t)\in\Delta^\delta}\Big|\frac{1}{\lfloor nt\rfloor - \lfloor ns\rfloor}\sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor}\dot\pi_j(\boldsymbol{U}_i)\,\Big[\breve{\mathbb{B}}_n^{(m)}\{s,t,\boldsymbol{h}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{U}_i)^{\{j\}}\} - \breve{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{U}_i^{\{j\}})\Big]\,\Big|.$$

For I_n', we have that

$$I_n' \le \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d}\big|\breve{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u})\big| \times \sup_{(s,t,\boldsymbol{u})\in\Delta^\delta\times[0,1]^d}\big|\dot\pi_j\{\boldsymbol{h}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u})\} - \dot\pi_j(\boldsymbol{u})\big| \xrightarrow{\mathrm{P}} 0$$

as a consequence of the weak convergence of $\breve{\mathbb{B}}_n^{(m)}$, (A.3), and the continuous mapping theorem. For I_n'', using the fact that $\sup_{\boldsymbol{u}\in[0,1]^d}|\dot\pi_j(\boldsymbol{u})|\le 1$, we obtain that

$$I_n'' \le \sup_{(s,t,\boldsymbol{u})\in\Delta^\delta\times[0,1]^d}\big|\breve{\mathbb{B}}_n^{(m)}\{s,t,\boldsymbol{h}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u})^{\{j\}}\} - \breve{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}^{\{j\}})\big| \xrightarrow{\mathrm{P}} 0.$$

The latter convergence is a consequence of the asymptotic equicontinuity in probability of $\breve{\mathbb{B}}_n^{(m)}$ and the fact that $\sup_{(s,t,u)\in\Delta^\delta\times[0,1]}|H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u) - u| \xrightarrow{\mathrm{P}} 0$ (see e.g. the treatment of the term (B.9) in Bücher et al., 2014, for a detailed proof of a similar convergence). ■

Proof of Proposition 3.3. We only provide the proof under (ii) in the statement of Proposition 3.1, the proof being simpler under (i). From Proposition 3.2, to prove the desired result it suffices to show that, for any $A\subseteq D$, $|A|\ge 1$,

$$\sup_{(s,t)\in\Delta}|\tilde{\mathbb{S}}_{n,b_n,A}^{(m)}(s,t) - \breve{\mathbb{S}}_{n,A}^{(m)}(s,t)| \xrightarrow{\mathrm{P}} 0.$$

Fix $A\subseteq D$, $|A|\ge 1$. From (3.4) and (3.5) and the triangle inequality, the latter will hold if, for any $j\in A$,

$$\sup_{(s,t,u)\in\Delta\times[0,1]}|\tilde{\mathbb{B}}_{n,b_n,j}^{(m)}(s,t,u) - \breve{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}_j)| \xrightarrow{\mathrm{P}} 0.$$

The previous supremum can actually be restricted to $u \in (0,1)$ as both processes are zero if $u \in \{0,1\}$.

Let $K > 0$ be a constant and let us first suppose that, for any $n \geq 1$ and $i \in \{1,\ldots,n\}$, $\xi_{i,n}^{(m)} \geq -K$. Also, fix $j \in A$. The supremum on the right of the previous display is then smaller than $I_n + II_n$, where

$$I_n = \sup_{(s,t,u)\in\Delta\times(0,1)} \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (\xi_{i,n}^{(m)} + K) \left| \mathcal{L}_{b_n}(\hat{U}_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}, u) - \mathbf{1}(\hat{U}_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \leq u) \right|,$$

$$II_n = \sup_{(s,t,u)\in\Delta\times(0,1)} \frac{K + \bar{\xi}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{(m)}}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left| \mathcal{L}_{b_n}(\hat{U}_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}, u) - \mathbf{1}(\hat{U}_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \leq u) \right|.$$

Next, some thought reveals that, for any $(u,v) \in [0,1] \times (0,1)$,

$$\begin{aligned} |\mathcal{L}_{b_n}(u,v) - \mathbf{1}(u \leq v)| &\leq \mathbf{1}(u_- \leq v) - \mathbf{1}(u_+ \leq v) \\ &= \mathbf{1}(u - b_n \leq v) - \mathbf{1}(u + b_n \leq v) \\ &= \mathbf{1}(u \leq v_+) - \mathbf{1}(u \leq v_-). \end{aligned} \tag{C.1}$$

Then, we write $I_n \leq I_{n,1} + I_{n,2}$, where

$$I_{n,1} = \sup_{(s,t,u)\in\Delta\times[0,1]} \left| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} (\xi_{i,n}^{(m)} - \bar{\xi}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{(m)}) \mathbf{1}(u_- < \hat{U}_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \leq u_+) \right|,$$

$$I_{n,2} = \sup_{(s,t,u)\in\Delta\times[0,1]} \frac{K + \bar{\xi}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}^{(m)}}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \mathbf{1}(u_- < \hat{U}_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \leq u_+).$$

For $I_{n,1}$, we have

$$I_{n,1} \leq \sup_{\substack{(s,t,\boldsymbol{u},\boldsymbol{v})\in\Delta\times[0,1]^{2d} \\ \|\boldsymbol{u}-\boldsymbol{v}\|_1 \leq 2b_n}} \left| \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) - \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{v}) \right| \xrightarrow{\mathrm{P}} 0$$

from the asymptotic uniform equicontinuity in probability of $\check{\mathbb{B}}_n^{(m)}$. Before dealing with $I_{n,2}$, let us first show that

$$I_{n,3} = \sup_{(s,t,u)\in\Delta\times[0,1]} \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \mathbf{1}(u_- < \hat{U}_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \leq u_+) \xrightarrow{\mathrm{P}} 0. \tag{C.2}$$

From the proof of Proposition 3.3 of Bücher et al. (2014), we have that

$$\sup_{(s,t,u)\in\Delta\times[0,1]} \left| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left[\mathbf{1}\{U_{ij} \leq H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}^{-1}(u)\} - \mathbf{1}(\hat{U}_{ij}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \leq u) \right] \right| \xrightarrow{\mathrm{P}} 0.$$

Consequently, to prove that $I_{n,3} \xrightarrow{\mathrm{P}} 0$, it suffices to show that

$$\sup_{(s,t,u)\in\Delta\times[0,1]} \left| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left[\mathbf{1}\{U_{ij} \leq H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}^{-1}(u_+)\} - \mathbf{1}\{U_{ij} \leq H_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}^{-1}(u_-)\} \right] \right| \xrightarrow{\mathrm{P}} 0.$$

The supremum on the left of the previous display is smaller than $J_{n,1} + J_{n,2} + J_{n,3}$, where

$$J_{n,1} = \sup_{(s,t,u)\in\Delta\times[0,1]} \left| \mathbb{B}_n\{s,t,1,H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u_+),1\} - \mathbb{B}_n\{s,t,1,H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u_-),1\} \right|,$$

$$J_{n,2} = \sup_{(s,t,u)\in\Delta\times[0,1]} \sqrt{n}\lambda_n(s,t) \left| H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u_+) - u_+ - H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u_-) + u_- \right|,$$

$$J_{n,3} = \sup_{(s,t,u)\in\Delta\times[0,1]} \sqrt{n}\lambda_n(s,t) \left| u_+ - u_- \right|,$$

with some abuse of notation for $J_{n,1}$. We immediately have $J_{n,3} \leq 2\sqrt{n}b_n \to 0$. The fact $J_{n,2} \xrightarrow{\mathrm{P}} 0$ follows from the asymptotic uniform equicontinuity in probability of the process $(s,t,u) \mapsto \sqrt{n}\lambda_n(s,t)\{H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u) - u\}$, itself following from its weak convergence to $(s,t,u) \mapsto -\mathbb{B}_C(s,t,\boldsymbol{u}_j)$ in $\ell^\infty(\Delta\times[0,1])$. The latter is a consequence of the weak convergence of $\mathbb{B}_n$ to $\mathbb{B}_C$ in $\ell^\infty(\Delta\times[0,1]^d)$, Lemma B.2 of Bücher and Kojadinovic (2013) and the extended continuous mapping theorem (van der Vaart and Wellner, 2000, Theorem 1.11.1). The fact that $J_{n,2} \xrightarrow{\mathrm{P}} 0$ implies that, for any $\delta \in (0,1)$,

$$\sup_{\substack{(s,t,u)\in\Delta\times[0,1]\\ t-s\geq\delta}} \left| H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u_+) - H^{-1}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor,j}(u_-) \right| \xrightarrow{\mathrm{P}} 0.$$

Combined with the asymptotic uniform equicontinuity in probability of $\mathbb{B}_n$, the latter can be used to prove that $J_{n,1} \xrightarrow{\mathrm{P}} 0$ (see Bücher et al., 2014, page 24, term (B.9), for a similar proof). Hence, $I_{n,3} \xrightarrow{\mathrm{P}} 0$.

Now, $I_{n,2} \leq K \times I_{n,3} + I_{n,4}$, where

$$I_{n,4} = \sup_{(s,t,u)\in\Delta\times[0,1]} \frac{\bar{\xi}^{(m)}_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \mathbf{1}(u_- < \hat{U}^{\lfloor ns\rfloor+1:\lfloor nt\rfloor}_{ij} \leq u_+).$$

Hence, to show that $I_{n,2} \xrightarrow{\mathrm{P}} 0$, it remains to prove that $I_{n,4} \xrightarrow{\mathrm{P}} 0$. The latter can be shown by proceeding as for the term (B.8) in Bücher et al. (2014).

We therefore have that $I_n \xrightarrow{\mathrm{P}} 0$. The fact that $II_n \xrightarrow{\mathrm{P}} 0$, follows from the fact that $II_n \leq I_{n,2} \xrightarrow{\mathrm{P}} 0$. This completes the proof under the condition $\xi^{(m)}_{i,n} \geq -K$. To show that this condition is not necessary, we use the arguments employed at the end of the proof of Proposition 4.3 of Bücher et al. (2014). ■

D Proof of Propositions 3.4 and 3.5

Lemma D.1. *Assume that $\boldsymbol{U}_1,\ldots,\boldsymbol{U}_n$ is drawn from a strictly stationary sequence $(\boldsymbol{U}_i)_{i\in\mathbb{Z}}$ whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$, $a > 6$. Then, for any $A \subseteq D$, $|A| \geq 1$ and $j \in A$, $\mathbb{H}_{n,A,j} \rightsquigarrow \mathbb{H}_{A,j}$ in $\ell^\infty([0,1])$, where, for any $t \in [0,1]$, $\mathbb{H}_{n,A,j}(t) = n^{-1/2}\sum_{i=1}^n [Y_{i,A,j}(t) - \mathrm{E}\{Y_{1,A,j}(t)\}]$, $Y_{i,A,j}(t) = \prod_{l\in A\setminus\{j\}}(1-U_{il})\mathbf{1}(t \leq U_{ij})$, and $\mathbb{H}_{A,j}$ is a tight process.*

Proof. Fix $A \subseteq D$, $|A| \geq 1$ and $j \in A$. To simplify the notation, we write $\mathbb{H}_n$ instead of $\mathbb{H}_{n,A,j}$ and Y_i instead of $Y_{i,A,j}$ as we continue. To prove the desired result, we mostly adapt the arguments used in the proof of Proposition 2.11 of Dehling and Philipp (2002). From Theorem 2.1 in Kosorok (2008), two conditions are needed to obtain the desired weak convergence. The first condition (which is the weak convergence of the finite-dimensional distributions) is a consequence of Theorem 3.23 of Dehling and Philipp (2002) as $a > 6$ and $Y_i(t) \in [0,1]$ for all $t \in [0,1]$. To prove the second condition, we shall show that $\mathbb{H}_n$ is asymptotically $|\cdot|$-equicontinuous in probability. To do so, we shall first prove that, for any $\varepsilon, \delta > 0$, there exists a grid $0 = t_0 < t_1 < \cdots < t_k = 1$ such that, for all n sufficiently large,

$$\mathrm{P}\left\{ \max_{1\leq i\leq k} \sup_{t\in[t_{i-1},t_i]} |\mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1})| \geq \varepsilon \right\} \leq \delta. \tag{D.1}$$

We first note that there exists constants $c \geq 1$ and $\epsilon \in (0,1)$ such that $\alpha_r \leq cr^{-6-\epsilon}$. Then, using the fact that, for $t, t' \in [0,1]$,

$$\mathrm{E}[\{Y_1(t) - Y_1(t')\}^2] \leq \mathrm{E}[|Y_1(t) - Y_1(t')|] \leq \mathrm{E}\{\mathbf{1}(t \wedge t' \leq U_{ij} \leq t \vee t')\} = |t - t'|,$$

we apply Lemma 3.22 of Dehling and Philipp (2002) with $\xi_i = Y_i(t) - Y_i(t')$ to obtain that

$$\mathrm{E}[\{\mathbb{H}_n(t) - \mathbb{H}_n(t')\}^4] \leq 10^4 \frac{c}{\epsilon}\left(|t-t'|^{\eta} + n^{-1}|t-t'|^{\eta/2}\right) = \lambda\left(|t-t'|^{\eta} + n^{-1}|t-t'|^{\eta/2}\right),$$

where $\eta = 1 + \epsilon/10 > 1$ and $\lambda = 10^4 c/\epsilon$. It follows that, for any $t, t' \in [0,1]$ such that $|t - t'| \geq n^{-2/\eta}$,

$$\mathrm{E}[\{\mathbb{H}_n(t) - \mathbb{H}_n(t')\}^4] \leq 2\lambda|t-t'|^{\eta}. \tag{D.2}$$

Next, consider a grid $0 = t_0 < t_1 < \cdots < t_k = 1$ to be specified later. Furthermore, it can be verified that the function $G : t \mapsto \mathrm{E}\{Y_1(t)\}$ is continuous and strictly decreasing on $[0,1]$. Then, fix $i \in \{1,\dots,k\}$, let $\tau = \varepsilon n^{-1/2}/4$, let $m = m_i = \lfloor \{G(t_{i-1}) - G(t_i)\}/\tau \rfloor$ and define a subgrid $t_{i-1} = s_0 < s_1 < \cdots < s_m = t_i$ such that $G(s_j) = G(s_0) - j\tau$ for $j \in \{1,\dots,m-1\}$. Notice that this ensures that, for any $j \in \{1,\dots,m\}$, $\tau \leq G(s_{j-1}) - G(s_j) \leq 2\tau$. Now, fix $j \in \{1,\dots,m\}$. Using the fact that the function $t \mapsto n^{-1}\sum_{i=1}^n Y_i(t)$ is also decreasing, it can be verified that, for any $t \in [s_{j-1}, s_j]$,

$$\mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1}) \leq |\mathbb{H}_n(s_{j-1}) - \mathbb{H}_n(t_{i-1})| + \varepsilon/2$$

and

$$-\varepsilon/2 - |\mathbb{H}_n(s_j) - \mathbb{H}_n(t_{i-1})| \leq \mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1}).$$

The above inequalities imply that, for any $t \in [t_{i-1}, t_i] = \bigcup_{j=1}^m [s_{j-1}, s_j]$,

$$-\varepsilon/2 + \min_{1\leq j\leq m}\{-|\mathbb{H}_n(s_j) - \mathbb{H}_n(t_{i-1})|\} \leq \mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1}) \leq \max_{2\leq j\leq m} |\mathbb{H}_n(s_{j-1}) - \mathbb{H}_n(t_{i-1})| + \varepsilon/2,$$

and thus that

$$\sup_{t\in[t_{i-1},t_i]} |\mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1})| \leq \max_{1\leq j\leq m} |\mathbb{H}_n(s_j) - \mathbb{H}_n(t_{i-1})| + \varepsilon/2.$$

Hence,

$$\mathrm{P}\left\{\sup_{t\in[t_{i-1},t_i]} |\mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1})| \geq \varepsilon\right\} \leq \mathrm{P}\left\{\max_{1\leq j\leq m} |\mathbb{H}_n(s_j) - \mathbb{H}_n(t_{i-1})| \geq \varepsilon/2\right\}. \tag{D.3}$$

Now, let $\zeta_l = \mathbb{H}_n(s_l) - \mathbb{H}_n(s_{l-1})$, $l \in \{1,\dots,m\}$ with $\zeta_0 = 0$, and let $S_j = \sum_{l=0}^{j} \zeta_l$, $j \in \{0,\dots,m\}$. From (D.2), we then have that, for any $0 \leq j < j' \leq m$ and n sufficiently large,

$$\mathrm{E}\{(S_{j'} - S_j)^4\} = \mathrm{E}\left\{\left(\sum_{l=j+1}^{j'} \zeta_l\right)^4\right\} = \mathrm{E}\left[\{\mathbb{H}_n(s_{j'}) - \mathbb{H}_n(s_j)\}^4\right]$$

$$\leq 2\lambda(s_{j'} - s_j)^\eta = 2\lambda\left\{\sum_{j<l\leq j'} (s_l - s_{l-1})\right\}^\eta.$$

Indeed, by construction of the subgrid, for any $0 \leq j < j' \leq m$, $n^{-1/2}\varepsilon/4 \leq G(s_j) - G(s_{j'}) \leq s_{j'} - s_j$, and $n^{-1/2}\varepsilon/4$ can be made larger than $n^{-2/\eta}$ by taking n sufficiently large since $2/\eta > 1/2$. The assumption of Theorem 2.12 of Billingsley (1968) being satisfied (see also Lemma 2.10 in Dehling and Philipp, 2002), we obtain that there exists a constant $K \geq 0$ such that, for any $\nu \geq 0$,

$$\mathrm{P}\left(\max_{1\leq j\leq m} |S_j| \geq \nu\right) \leq \nu^{-4}K(s_m - s_0)^\eta = \nu^{-4}K(t_i - t_{i-1})^\eta.$$

Applying the previous inequality to the right-hand side of (D.3), we obtain that

$$\mathrm{P}\left\{\sup_{t\in[t_{i-1},t_i]} |\mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1})| \geq \varepsilon\right\} \leq \varepsilon^{-4}2^4K(t_i - t_{i-1})^\eta.$$

It follows that

$$\mathrm{P}\left\{\max_{1\leq i\leq k} \sup_{t\in[t_{i-1},t_i]} |\mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1})| \geq \varepsilon\right\} \leq \varepsilon^{-4}2^4K\sum_{i=1}^{k}(t_i - t_{i-1})^\eta$$

$$\leq \varepsilon^{-4}2^4K \times \max_{1\leq i\leq k}(t_i - t_{i-1})^{\eta-1} \times \sum_{i=1}^{k}(t_i - t_{i-1}).$$

By choosing the initial grid such that $\max_{1\leq i\leq k}(t_i - t_{i-1}) \leq \{\delta\varepsilon^4 2^{-4}K^{-1}\}^{1/(\eta-1)}$, we obtain (D.1).

It remains to verify that $\mathbb{H}_n$ is asymptotically $|\cdot|$-equicontinuous in probability. By Problem 2.1.5 in van der Vaart and Wellner (2000), this amounts to showing that for any positive sequence $a_n \downarrow 0$ and any $\varepsilon, \delta > 0$,

$$\mathrm{P}\left\{\sup_{\substack{s,t\in[0,1]\\ |t-s|\leq a_n}} |\mathbb{H}_n(s) - \mathbb{H}_n(t)| > 3\varepsilon\right\} \leq \delta \tag{D.4}$$

for n sufficiently large. Fix $\varepsilon, \delta > 0$ and $a_n \downarrow 0$, and choose a grid $0 = t_0 < \cdots < t_k = 1$ such that (D.1) holds for all n sufficiently large. Furthermore, let $\mu = \min_{1<i<k}(t_i - t_{i-1})$. Then, from Billingsley (1999, Theorem 7.4), we have that, for all n sufficiently large such that $a_n \leq \mu$,

$$\sup_{\substack{s,t\in[0,1]\\ |t-s|\leq a_n}} |\mathbb{H}_n(s) - \mathbb{H}_n(t)| \leq 3 \max_{1\leq i\leq k} \sup_{t\in[t_{i-1},t_i]} |\mathbb{H}_n(t) - \mathbb{H}_n(t_{i-1})|.$$

Finally, (D.4) follows for all n sufficiently large by combining the previous inequality with (D.1). ■

Proof of Proposition 3.4. We shall only prove the result under (ii), the proof being simpler under (i). Recall $\sigma^2_{n,C,f}$ defined in (3.13). From (3.15), we immediately have that $\sigma^2_{n,C,f} \overset{\mathrm{P}}{\to} \sigma^2_{C,f}$. It remains to show that $\check{\sigma}^2_{n,C_{1:n},f} - \sigma^2_{n,C,f} \overset{\mathrm{P}}{\to} 0$.

Recall $\boldsymbol{h}_{1:n}$ defined in (A.1) and that $\hat{\boldsymbol{U}}_i^{1:n} = \boldsymbol{h}_{1:n}(\boldsymbol{U}_i)$ for all $i \in \{1,\ldots,n\}$. Then, starting from (3.10) and (3.14), it can be verified that

$$\begin{aligned}|\check{\sigma}^2_{n,C_{1:n},f} - \sigma^2_{n,C,f}| \leq &\left\{\frac{1}{n}\sum_{i,j=1}^{n} \varphi\left(\frac{i-j}{\ell_n}\right)\right\} \\ &\times \left[\sup_{\boldsymbol{u}\in[0,1]^d} |f\{\mathcal{I}_C(\boldsymbol{u}) - \psi_C(C)\}| + \sup_{\boldsymbol{u}\in[0,1]^d} |f[\mathcal{I}_{C_{1:n}}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \psi_{C_{1:n}}(C_{1:n})]|\right] \\ &\times \sup_{\boldsymbol{u}\in[0,1]^d} |f[\mathcal{I}_{C_{1:n}}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_C(\boldsymbol{u}) - \psi_{C_{1:n}}(C_{1:n}) + \psi_C(C)]|. \quad \text{(D.5)}\end{aligned}$$

Some algebra shows that the second term on the right of the previous inequality is smaller than

$$\sup_{\boldsymbol{u}\in[0,1]^d} |f\circ\mathcal{I}_C(\boldsymbol{u})| + |f\circ\psi_C(C)| + 2\sup_{\boldsymbol{u}\in[0,1]^d} |f\circ\mathcal{I}_{C_{1:n}}(\boldsymbol{u})|.$$

From (3.2) and (2.14), we have that, for any $A \subseteq D$, $|A| \geq 1$, $\sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C,A}(\boldsymbol{u})| \leq 1$, $\sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C_{1:n},A}(\boldsymbol{u})| \leq 1$ and $|\psi_{C,A}(C)| \leq 1$. Hence, by (2.17), (3.8) and linearity of f, we have that the second term (between square brackets) on the right of inequality (D.5) is bounded by $4\sup_{\boldsymbol{x}\in[-1,1]^{2^d-1}} |f(\boldsymbol{x})| < \infty$. Concerning the first term on the right of (D.5), we have

$$\frac{1}{n}\sum_{i,j=1}^{n} \varphi\left(\frac{i-j}{\ell_n}\right) = \frac{1}{n}\sum_{k=-\ell_n}^{\ell_n} (n-|k|)\varphi\left(\frac{k}{\ell_n}\right) \leq 2\ell_n + 1 = O(n^{1/2-\varepsilon}).$$

We will now show that the last supremum on the right of (D.5) is $O_{\mathrm{P}}(n^{-1/2})$, which will complete the proof. By the triangle inequality,

$$\begin{aligned}&\sup_{\boldsymbol{u}\in[0,1]^d} |f[\mathcal{I}_{C_{1:n}}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_C(\boldsymbol{u}) - \psi_{C_{1:n}}(C_{1:n}) + \psi_C(C)]| \\ &\qquad\qquad \leq \sup_{\boldsymbol{u}\in[0,1]^d} |f[\mathcal{I}_{C_{1:n}}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_C(\boldsymbol{u})]| + |f\{\psi_{C_{1:n}}(C_{1:n}) - \psi_C(C)\}|.\end{aligned}$$

By linearity of f, from (3.2) and (3.8), to show that the first term on the right on the previous inequality is $O_{\mathrm{P}}(n^{-1/2})$, it suffices to show that, for any $A \subseteq D$, $|A| \geq 1$,

$$\sup_{\boldsymbol{u} \in [0,1]^d} |\mathcal{I}_{C_{1:n},A}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_{C,A}(\boldsymbol{u})| = O_{\mathrm{P}}(n^{-1/2}). \tag{D.6}$$

Similarly, for the second term on the right, it suffices to show that, for any $A \subseteq D$, $|A| \geq 1$, $|\psi_{C_{1:n},A}(C_{1:n}) - \psi_{C,A}(C)| = O_{\mathrm{P}}(n^{-1/2})$. Now, from Fubini's theorem, $\psi_{C,A}(C) = \psi_{C,A}[\mathrm{E}\{\mathbf{1}(\boldsymbol{U}_1 \leq \cdot)\}] = \mathrm{E}\{\mathcal{I}_{C,A}(\boldsymbol{U}_1)\}$. Hence, $|\psi_{C_{1:n},A}(C_{1:n}) - \psi_{C,A}(C)|$ is smaller than

$$\begin{aligned} &\left| \frac{1}{n} \sum_{i=1}^n \left\{ \mathcal{I}_{C_{1:n},A}(\hat{\boldsymbol{U}}_i^{1:n}) - \mathcal{I}_{C,A}(\boldsymbol{U}_i) \right\} \right| + \left| \frac{1}{n} \sum_{i=1}^n [\mathcal{I}_{C,A}(\boldsymbol{U}_i) - \mathrm{E}\{\mathcal{I}_{C,A}(\boldsymbol{U}_1)\}] \right| \\ &\qquad \leq \sup_{\boldsymbol{u} \in [0,1]^d} |\mathcal{I}_{C_{1:n},A}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_{C,A}(\boldsymbol{u})| + \left| \frac{1}{n} \sum_{i=1}^n [\mathcal{I}_{C,A}(\boldsymbol{U}_i) - \mathrm{E}\{\mathcal{I}_{C,A}(\boldsymbol{U}_1)\}] \right|. \end{aligned}$$

The proof is therefore complete if we show (D.6) and that the second term on the right of the previous inequality is $O_{\mathrm{P}}(n^{-1/2})$. The latter is a consequence of the weak convergence of $n^{-1/2} \sum_{i=1}^n [\mathcal{I}_{C,A}(\boldsymbol{U}_i) - \mathrm{E}\{\mathcal{I}_{C,A}(\boldsymbol{U}_1)\}]$ which follows from Theorem 3.23 of Dehling and Philipp (2002) as a consequence of the fact that $\sup_{\boldsymbol{u} \in [0,1]^d} |\mathcal{I}_{C,A}(\boldsymbol{u})| \leq 1$ and the assumption on the mixing rate.

It remains to prove (D.6). The latter will follow by the triangle inequality if we show that, for any $A \subseteq D$, $|A| \geq 1$,

$$\sup_{\boldsymbol{u} \in [0,1]^d} |\mathcal{I}_{C,A}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_{C,A}(\boldsymbol{u})| = O_{\mathrm{P}}(n^{-1/2}), \tag{D.7}$$

$$\sup_{\boldsymbol{u} \in [0,1]^d} |\mathcal{I}_{H_{1:n},A}(\boldsymbol{u}) - \mathcal{I}_{C,A}(\boldsymbol{u})| = O_{\mathrm{P}}(n^{-1/2}), \tag{D.8}$$

$$\sup_{\boldsymbol{u} \in [0,1]^d} |\mathcal{I}_{C_{1:n},A}(\boldsymbol{u}) - \mathcal{I}_{H_{1:n},A}(\boldsymbol{u})| = O_{\mathrm{P}}(n^{-1/2}). \tag{D.9}$$

Fix $A \subseteq D$, $|A| \geq 1$.

Proof of (D.7). We have

$$\begin{aligned} \sup_{\boldsymbol{u} \in [0,1]^d} |\mathcal{I}_{C,A}\{\boldsymbol{h}_{1:n}(\boldsymbol{u})\} - \mathcal{I}_{C,A}(\boldsymbol{u})| &\leq \sup_{\boldsymbol{u} \in [0,1]^d} \left| \prod_{l \in A} \{1 - H_{1:n,l}(u_l)\} - \prod_{l \in A} (1 - u_l) \right| \\ &+ \sum_{j \in A} \sup_{u \in [0,1]} \left| \int_{[0,1]^d} \prod_{l \in A \setminus \{j\}} (1 - v_l) \, [\mathbf{1}\{H_{1:n,j}(u) \leq v_j\} - \mathbf{1}(u \leq v_j)] \, \mathrm{d}C(\boldsymbol{v}) \right|. \end{aligned}$$

By an application of the mean value theorem similar to that performed in the proof of Proposition 2.1, it is easy to verify that the first supremum is $O_{\mathrm{P}}(n^{-1/2})$ since, for any $j \in D$, $\sup_{u \in [0,1]} |H_{1:n,j}(u) - u| = O_{\mathrm{P}}(n^{-1/2})$ as a consequence of the weak convergence of

$\mathbb{B}_n$ defined in (2.10). The second term is smaller than

$$\sum_{j\in A} \sup_{u\in[0,1]} \int_{[0,1]} |\mathbf{1}\{H_{1:n,j}(u) \le v\} - \mathbf{1}(u \le v)| \,\mathrm{d}v$$

$$\le \sum_{j\in A} \sup_{u\in[0,1]} \int_{[0,1]} \mathbf{1}\{u \wedge H_{1:n,j}(u) \le v \le u \vee H_{1:n,j}(u)\}\mathrm{d}v$$

$$= \sum_{j\in A} \sup_{u\in[0,1]} |H_{1:n,j}(u) - u| = O_{\mathrm{P}}(n^{-1/2}).$$

Proof of (D.8)*:* From (3.2) and the triangle inequality, it suffices to show that, for any $j \in A$,

$$\sup_{u\in[0,1]} \left| \frac{1}{n} \sum_{i=1}^{n} \prod_{l\in A\setminus\{j\}} (1 - U_{il})\mathbf{1}(u \le U_{ij}) - \int_{[0,1]^d} \prod_{l\in A\setminus\{j\}} (1 - v_l)\mathbf{1}(u \le v_j)\mathrm{d}C(\boldsymbol{v}) \right| = O_{\mathrm{P}}(n^{-1/2}).$$

The latter is an immediate consequence of the weak convergence result stated in Lemma D.1 and the continuous mapping theorem.

Proof of (D.9)*:* The supremum on the left of (D.9) is smaller than $I_n + II_n + III_n$, where

$$I_n = \sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C_{1:n},A}(\boldsymbol{u}) - \mathcal{I}_{H_{1:n},A}\{\boldsymbol{h}_{1:n}^{-1}(\boldsymbol{u})\}|,$$

$$II_n = \sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{H_{1:n},A}\{\boldsymbol{h}_{1:n}^{-1}(\boldsymbol{u})\} - \mathcal{I}_{C,A}\{\boldsymbol{h}_{1:n}^{-1}(\boldsymbol{u})\} - \mathcal{I}_{H_{1:n},A}(\boldsymbol{u}) + \mathcal{I}_{C,A}(\boldsymbol{u})|, \tag{D.10}$$

$$III_n = \sup_{\boldsymbol{u}\in[0,1]^d} |\mathcal{I}_{C,A}\{\boldsymbol{h}_{1:n}^{-1}(\boldsymbol{u})\} - \mathcal{I}_{C,A}(\boldsymbol{u})|, \tag{D.11}$$

with $\boldsymbol{h}_{1:n}^{-1}$ is defined in (A.2). The term I_n is smaller

$$\sup_{\boldsymbol{u}\in(0,1]^d} \left| \prod_{l\in A}(1 - u_l) - \prod_{l\in A}\{1 - H_{1:n,l}^{-1}(u_l)\} \right|$$

$$+ \sup_{\boldsymbol{u}\in[0,1]^d} \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{j\in A} \prod_{l\in A\setminus\{j\}} \{1 - H_{1:n,l}(U_{il})\}\mathbf{1}\{u_j \le H_{1:n,j}(U_{ij})\} \right.$$

$$\left. - \frac{1}{n} \sum_{i=1}^{n} \sum_{j\in A} \prod_{l\in A\setminus\{j\}} (1 - U_{il})\mathbf{1}\{H_{1:n,j}^{-1}(u_j) \le U_{ij}\} \right|.$$

Since, for any $j \in D$, $\sup_{u\in[0,1]} |H_{1:n,j}^{-1}(u) - u| = \sup_{u\in[0,1]} |H_{1:n,j}(u) - u|$ (for instance, by symmetry arguments on the graphs of $H_{1:n,j}$ and $H_{1:n,j}^{-1}$), and by an application of the mean value theorem as above, we obtain that the first supremum is $O_{\mathrm{P}}(n^{-1/2})$. Using the fact that, for all $u \in [0,1]$, $u \le H_{1:n,j}(U_{ij})$ is equivalent to $H_{1:n,j}^{-1}(u) \le U_{ij}$, it can be verified that the second supremum is smaller than

$$\sum_{j\in A}\sup_{u\in[0,1]}\left|\frac{1}{n}\sum_{i=1}^{n}\left[\prod_{l\in A\setminus\{j\}}\{1-H_{1:n,l}(U_{il})\}-\prod_{l\in A\setminus\{j\}}(1-U_{il})\right]\mathbf{1}\{u\leq H_{1:n,j}(U_{ij})\}\right|$$
$$\leq\sum_{j\in A}\sup_{\boldsymbol{u}\in[0,1]^d}\left|\prod_{l\in A\setminus\{j\}}\{1-H_{1:n,l}(u_l)\}-\prod_{l\in A\setminus\{j\}}(1-u_l)\right|=O_{\mathrm{P}}(n^{-1/2}),$$

where the last equality follows again by an application of the mean value theorem as above. Hence, $I_n=O_{\mathrm{P}}(n^{-1/2})$. For II_n defined in (D.10), we have

$$II_n\leq n^{-1/2}\sum_{j\in A}\sup_{u\in[0,1]}\left|\mathbb{H}_{n,A,j}\{H_{1:n,j}^{-1}(u)\}-\mathbb{H}_{n,A,j}(u)\right|=o_{\mathrm{P}}(n^{-1/2}),$$

where $\mathbb{H}_{n,A,j}$ is defined in Lemma D.1. The last equality is a consequence of the asymptotic equicontinuity in probability of $\mathbb{H}_{n,A,j}$ and the fact that $\sup_{u\in[0,1]}|H_{1:n,j}^{-1}(u)-u|=\sup_{u\in[0,1]}|H_{1:n,j}(u)-u|\xrightarrow{\text{a.s.}}0$. The latter convergence follows from the almost sure invariance principle established in Berkes and Philipp (1977) and Yoshihara (1979). It implies a functional law of the iterated logarithm for $u\mapsto H_{1:n,j}(u)-u$ as soon as $a>3$, which in turn implies the Glivenko–Cantelli lemma under strong mixing.

It remains to show that III_n defined in (D.11) is $O_{\mathrm{P}}(n^{-1/2})$. The proof of the latter is similar to that of (D.7). ■

Proof of Proposition 3.5. We only show the result under (ii), the proof being simpler under (i). To prove the desired result, we shall show that $\tilde{\sigma}^2_{n,b_n,C_{1:n},f}-\check{\sigma}^2_{n,C_{1:n},f}\xrightarrow{\mathrm{P}}0$. Proceeding as in the proof of Proposition 3.4 for (D.5), it can be verified that to prove the above, it suffices to show that, for any $A\subseteq D$, $|A|\geq 1$,

$$\sup_{\boldsymbol{u}\in[0,1]^d}|\mathcal{I}_{b_n,C_{1:n},A}(\boldsymbol{u})-\mathcal{I}_{C_{1:n},A}(\boldsymbol{u})|=O_{\mathrm{P}}(n^{-1/2}).$$

Fix $A\subseteq D$, $|A|\geq 1$. From (3.2) and (3.6), we have that the supremum on the right of the previous display is smaller than $\sum_{j\in A}I_{n,j}$, where

$$I_{n,j}=\sup_{u\in[0,1]}\int_{[0,1]^d}|\mathcal{L}_{b_n}(u,v_j)-\mathbf{1}(u\leq v_j)|\,\mathrm{d}C_{1:n}(\boldsymbol{v}).$$

Fix $j\in A$. From (C.1), we have that $I_{n,j}\leq n^{-1/2}J_{n,j}$, where

$$\begin{aligned}J_{n,j}&=\sup_{u\in[0,1]}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{\mathbf{1}(u_-\leq\hat{U}_{ij}^{1:n})-\mathbf{1}(u_+\leq\hat{U}_{ij}^{1:n})\}\\&=\sup_{u\in[0,1]}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{\mathbf{1}(\hat{U}_{ij}^{1:n}<u_+)-\mathbf{1}(\hat{U}_{ij}^{1:n}<u_-)\}\\&\leq\sup_{u\in[0,1]}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{\mathbf{1}(\hat{U}_{ij}^{1:n}\leq u_+)-\mathbf{1}(\hat{U}_{ij}^{1:n}\leq u_-)\}+\sup_{u\in[0,1]}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{1}(\hat{U}_{ij}^{1:n}=u).\end{aligned}$$

Proceeding as for (C.2), we obtain that the first supremum on the right of the previous display converges in probability to zero. The second supremum is smaller than

$$\sup_{u\in[0,1]} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{\mathbf{1}(\hat{U}_{ij}^{1:n} \leq u) - \mathbf{1}(\hat{U}_{ij}^{1:n} \leq u - 1/n)\}$$

and can be dealt with along the same lines. Hence, $J_{n,j} \xrightarrow{\mathrm{P}} 0$, which implies that $I_{n,j} = o(n^{-1/2})$ and completes the proof. ■

References

I. Berkes and W. Philipp. An almost sure invariance principle for the empirical distribution function of mixing random variables. *Probability Theory and Related Fields*, 41: 115–137, 1977.

P. Billingsley. *Convergence of probability Measures.* Wiley, New York, 1968.

P. Billingsley. *Convergence of probability Measures.* Wiley, New York, 1999. Second edition.

A. Bücher. A note on weak convergence of the sequential multivariate empirical process under strong mixing. *Journal of Theoretical Probability*, page in press, 2013.

A. Bücher and I. Kojadinovic. A dependent multiplier bootstrap for the sequential empirical copula process under strong mixing. *arXiv:1306.3930*, 2013.

A. Bücher and J. Segers. Extreme value copula estimation based on block maxima of a multivariate stationary time series. *Extremes*, page in press, 2014.

A. Bücher, I. Kojadinovic, T. Rohmer, and J. Segers. Detecting changes in cross-sectional dependence in multivariate time series. *arXiv:1206.2557*, 2014.

P. Bühlmann. *The blockwise bootstrap in time series and empirical processes.* PhD thesis, ETH Zürich, 1993. Diss. ETH No. 10354.

M. Csörgő and L. Horváth. *Limit theorems in change-point analysis.* Wiley Series in Probability and Statistics. John Wiley & Sons, Chichester, UK, 1997.

R.M. de Jong and J. Davidson. Consistency of kernel estimators of heteroscedastic and autocorrelated covariance matrices. *Econometrica*, 68(2):407–423, 2000.

P. Deheuvels. A non parametric test for independence. *Publications de l'Institut de Statistique de l'Université de Paris*, 26:29–50, 1981.

H. Dehling and W. Philipp. Empirical process techniques for dependent data. In H. Dehling, T. Mikosch, and M. Sorensen, editors, *Empirical process techniques for dependent data*, pages 1–113. Birkhäuser, Boston, 2002.

H. Dehling, D. Vogel, M. Wendler, and D. Wied. Testing for changes in the rank correlation of time series. *arXiv:1203.4871*, 2014.

E. Gombay and L. Horváth. Change-points and bootstrap. *Environmetrics*, 10(6), 1999.

E. Gombay and L. Horváth. Rates of convergence for U-statistic processes and their bootstrapped versions. *Journal of Statistical Planning and Inference*, 102:247–272, 2002.

M. Hofert, I. Kojadinovic, M. Mächler, and J. Yan. *copula: Multivariate dependence with copulas*, 2013. URL `http://CRAN.R-project.org/package=copula`. R package version 0.999-7.

M. Holmes, I. Kojadinovic, and J-F. Quessy. Nonparametric tests for change-point detection à la Gombay and Horváth. *Journal of Multivariate Analysis*, 115:16–32, 2013.

I. Kojadinovic. *npcp: Some nonparametric tests for change-point detection in (multivariate) observations*, 2014. URL `http://CRAN.R-project.org/package=npcp`. R package version 0.0-1.

M.R. Kosorok. *Introduction to empirical processes and semiparametric inference.* Springer, New York, 2008.

H.R. Künsch. The jacknife and the bootstrap for general stationary observations. *Annals of Statistics*, 17(3):1217–1241, 1989.

R.B. Nelsen. *An introduction to copulas.* Springer, New-York, 2006. Second edition.

E. Paparoditis and D.N. Politis. Tapered block bootstrap. *Biometrika*, 88(4):1105–1119, 2001.

D.N. Politis and H. White. Automatic block-length selection for the dependent bootstrap. *Econometric Reviews*, 23(1):53–70, 2004.

D.P. Politis and J.P. Romano. Bias-corrected nonparametric spectral estimation. *Journal of Time Series Analysis*, 16:67–103, 1995.

J-F. Quessy. Theoretical efficiency comparisons of independence tests based on multivariate versions of Spearman's rho. *Metrika*, 70(3):315–338, 2009.

J.-F. Quessy, M. Saïd, and A.-C. Favre. Multivariate Kendall's tau for change-point detection in copulas. *The Canadian Journal of Statistics*, 41:65–82, 2013.

R Development Core Team. *R: A Language and Environment for Statistical Computing.* R Foundation for Statistical Computing, Vienna, Austria, 2014. URL `http://www.R-project.org`. ISBN 3-900051-07-0.

L. Rüschendorf. Asymptotic distributions of multivariate rank order statistics. *Annals of Statistics*, 4:912–923, 1976.

F. Schmid and R. Schmidt. Multivariate extensions of Spearman's rho and related statistics. *Statistics & Probability Letters*, 77(4):407–416, 2007.

X. Shao. The dependent wild bootstrap. *Journal of the American Statistical Association*, 105(489):218–235, 2010.

X. Shao and X. Zhang. Testing for change points in time series. *Journal of the American Statistical Association*, 105(491):1228–1240, 2010.

G.R. Shorack and J.A. Wellner. *Empirical processes with applications to statistics.* Wiley Series in Probability and Mathematical Statistics. John Wiley and Sons, New York, 1986.

A. Sklar. Fonctions de répartition à n dimensions et leurs marges. *Publications de l'Institut de Statistique de l'Université de Paris*, 8:229–231, 1959.

A.W. van der Vaart and J.A. Wellner. *Weak convergence and empirical processes.* Springer, New York, 2000. Second edition.

D. Wied, H. Dehling, M. van Kampen, and D. Vogel. A fluctuation test for constant Spearman's rho with nuisance-free limit distribution. *Computational Statistics and Data Analysis*, 76:723–736, 2014.

K. Yoshihara. Note on an almost sure invariance principle for some empirical processes. *Yokohama Mathematical Journal*, 27:105–110, 1979.

Table 1: Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{50, 100, 200, 400\}$ generated with $\beta = 0$ and when $C_1 = C_2 = C$ is either the d-dimensional Clayton (Cl) or Gumbel–Hougaard (GH) copula the bivariate margins of which have a Kendall's tau of τ. The tests $\tilde{S}_{n,i}$ are carried out with i.i.d. multiplier sequences, while the tests $S^a_{n,i}$ use variance estimators of the form (3.11).

			$d = 2$				$d = 4$					
C	n	τ	$\tilde{S}_{n,1}$	$\tilde{S}_{n,2}$	$S^a_{n,1}$	$S^a_{n,2}$	$\tilde{S}_{n,1}$	$\tilde{S}_{n,2}$	$\tilde{S}_{n,3}$	$S^a_{n,1}$	$S^a_{n,2}$	$S^a_{n,3}$
Cl	50	0.1	6.8	7.4	2.6	3.0	4.6	5.1	4.0	1.2	2.1	0.7
		0.3	4.1	5.2	1.7	4.2	4.9	5.4	3.7	0.5	2.6	0.7
		0.5	3.1	2.7	2.5	8.6	7.1	3.9	4.9	2.8	2.8	1.2
		0.7	3.0	0.5	8.3	23.8	7.4	4.1	3.3	5.4	10.3	3.1
	100	0.1	3.5	4.3	2.3	2.7	4.1	5.3	4.4	1.6	3.4	2.5
		0.3	4.0	4.4	2.3	3.6	5.7	4.7	4.4	2.0	2.8	1.4
		0.5	4.2	4.0	4.9	8.3	4.3	4.0	3.5	2.2	3.7	1.9
		0.7	5.7	1.6	12.6	23.1	9.1	3.9	7.6	11.3	9.5	7.4
	200	0.1	4.9	4.7	2.8	3.1	6.1	5.1	5.2	3.1	3.4	3.3
		0.3	4.9	5.3	3.7	4.9	4.1	5.6	4.2	2.3	3.6	1.9
		0.5	4.6	4.3	4.8	6.9	4.6	5.5	4.2	4.1	4.8	3.2
		0.7	5.6	3.1	11.2	15.1	10.5	5.3	11.1	14.1	8.3	9.9
	400	0.1	4.6	4.9	3.7	3.8	6.3	6.7	6.5	4.5	5.5	4.8
		0.3	4.3	4.6	4.0	4.4	5.8	5.3	5.5	4.1	4.2	3.8
		0.5	4.8	4.6	4.2	4.8	5.8	4.5	5.5	5.5	4.0	4.7
		0.7	5.9	4.0	9.3	10.8	8.5	6.6	8.7	13.5	8.1	8.2
GH	50	0.1	6.7	6.3	3.4	2.3	5.8	5.3	4.7	2.4	0.8	2.5
		0.3	4.1	3.9	3.5	2.1	5.9	6.0	5.3	1.8	0.7	3.1
		0.5	3.1	3.4	6.9	3.4	4.6	4.9	4.0	3.0	2.5	6.5
		0.7	2.0	1.8	15.5	10.7	3.4	6.2	2.0	6.2	4.2	10.3
	100	0.1	5.2	5.1	2.7	2.5	4.3	4.8	4.1	2.5	1.5	2.1
		0.3	5.9	5.3	5.2	3.9	6.1	6.7	6.7	3.1	1.9	4.5
		0.5	3.7	3.7	6.6	5.1	5.3	4.8	5.3	3.6	3.4	6.4
		0.7	1.3	2.3	16.9	13.8	4.5	7.0	2.7	8.6	9.0	14.2
	200	0.1	5.2	5.2	3.8	3.5	4.8	4.3	4.5	3.3	2.6	3.1
		0.3	5.2	5.1	4.7	3.9	6.0	6.5	5.3	4.7	3.3	4.3
		0.5	4.5	4.5	5.2	4.7	4.2	3.9	4.0	3.2	3.6	3.9
		0.7	2.2	3.7	12.8	10.8	4.6	7.0	4.9	6.6	9.0	10.9
	400	0.1	6.4	6.1	4.8	4.7	5.1	5.7	4.3	4.0	3.1	3.1
		0.3	4.7	4.6	4.1	3.8	4.6	5.3	5.6	3.7	3.6	4.4
		0.5	3.3	3.3	3.5	3.0	4.3	5.1	4.5	3.9	4.5	4.7
		0.7	4.6	5.8	10.1	9.9	5.3	7.1	5.9	6.3	9.5	10.4

Table 2: Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{50, 100, 200\}$ generated with $\beta = 0$, $t \in \{0.1, 0.25, 0.5\}$ and when C_1 and C_2 are both d-dimensional normal (N) or Frank (F) copulas such that the bivariate margins of C_1 have a Kendall's tau of 0.2 and those of C_2 a Kendall's tau of τ. The colunms CvM give the results for the test studied in Bücher et al. (2014). All the tests were carried out with i.i.d. multiplier sequences.

				$d = 2$			$d = 4$			
C	n	τ	t	CvM	$\tilde{S}_{n,1}$	$\tilde{S}_{n,2}$	CvM	$\tilde{S}_{n,1}$	$\tilde{S}_{n,2}$	$\tilde{S}_{n,3}$
N	50	0.4	0.10	5.6	6.0	5.6	5.9	7.9	7.9	8.3
			0.25	9.1	8.7	8.9	12.2	17.3	18.9	19.5
			0.50	13.4	12.6	12.6	24.3	25.1	27.6	28.2
		0.6	0.10	9.0	8.7	8.9	7.1	20.7	21.7	22.4
			0.25	32.3	34.7	32.6	45.6	66.3	67.0	69.9
			0.50	46.7	42.7	41.6	76.1	78.0	77.5	80.8
	100	0.4	0.10	5.7	7.8	7.6	7.6	11.2	12.2	12.3
			0.25	14.9	19.7	19.1	27.0	35.3	37.2	43.0
			0.50	25.9	28.9	29.2	54.5	54.6	53.5	59.6
		0.6	0.10	14.6	22.7	23.4	26.1	47.5	51.1	58.8
			0.25	60.0	68.6	69.0	90.3	94.9	94.8	97.6
			0.50	81.9	84.8	84.2	98.8	98.4	99.0	99.5
	200	0.4	0.10	9.1	11.7	12.3	13.2	18.2	17.9	23.3
			0.25	26.5	36.7	36.9	58.9	64.9	67.1	75.5
			0.50	47.7	54.2	53.7	83.4	83.5	83.3	88.9
		0.6	0.10	34.5	57.7	58.0	63.1	87.3	87.8	93.8
			0.25	92.6	96.5	96.7	100.0	100.0	100.0	100.0
			0.50	99.1	99.5	99.5	100.0	100.0	100.0	100.0
F	50	0.4	0.10	6.9	5.7	6.2	4.5	7.8	9.0	8.4
			0.25	10.8	9.7	10.0	12.9	17.9	19.7	19.9
			0.50	15.1	13.6	13.6	24.7	30.2	31.1	29.1
		0.6	0.10	11.1	10.6	11.3	7.3	23.3	29.7	24.8
			0.25	33.1	32.7	31.9	42.3	67.2	70.2	69.5
			0.50	50.9	46.1	46.2	78.3	81.9	82.3	85.5
	100	0.4	0.10	6.1	7.0	7.4	6.5	9.2	13.6	11.9
			0.25	16.5	18.2	18.7	26.5	38.8	46.8	49.6
			0.50	26.4	28.6	28.3	48.9	52.7	58.3	61.6
		0.6	0.10	17.7	27.3	27.2	22.7	55.3	63.9	68.6
			0.25	66.5	73.6	74.0	91.9	97.7	98.2	99.5
			0.50	86.2	87.3	87.5	99.3	98.8	99.4	99.8
	200	0.4	0.10	10.2	15.7	15.6	12.5	19.7	25.3	27.1
			0.25	34.3	41.3	41.5	53.6	64.4	76.2	78.8
			0.50	50.7	54.3	54.4	83.2	83.9	90.4	93.2
		0.6	0.10	39.0	64.7	65.6	60.3	88.0	92.2	96.4
			0.25	95.4	98.3	98.3	99.9	100.0	100.0	100.0
			0.50	99.5	99.8	99.8	100.0	100.0	100.0	100.0

Table 3: Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{100, 200, 400\}$ when $C_1 = C_2 = C$ is either the bivariate Clayton (Cl), Gumbel–Hougaard (GH) or Frank (F) copula with a Kendall's tau of τ. In the first three vertical blocks of the table, the test $\tilde{S}_{n,1}$ (resp. $S^a_{n,1}$) is carried out using dependent multiplier sequences (resp. a variance estimator of the form (3.12)). In the last vertical block, i.i.d. multipliers and a variance estimator of the form (3.11) are used instead.

			$\beta = 0$		$\beta = 0.25$		$\beta = 0.5$		$\beta = 0.5/\text{ind}$	
C	n	τ	$\tilde{S}_{n,1}$	$S^a_{n,1}$	$\tilde{S}_{n,1}$	$S^a_{n,1}$	$\tilde{S}_{n,1}$	$S^a_{n,1}$	$\tilde{S}_{n,1}$	$S^a_{n,1}$
Cl	100	0.10	5.2	2.3	6.6	3.5	8.2	3.3	14.5	10.2
		0.30	3.5	1.8	6.7	3.1	7.1	4.7	15.0	11.6
		0.50	4.0	3.4	5.0	4.5	5.2	4.7	12.0	13.5
		0.70	8.3	12.0	7.5	11.8	7.2	11.2	8.9	20.0
	200	0.10	4.2	2.3	5.1	2.8	6.9	3.6	17.2	13.5
		0.30	5.1	2.6	6.2	3.4	7.2	4.4	15.7	13.0
		0.50	4.4	4.1	5.0	5.1	4.6	5.1	14.1	14.2
		0.70	6.5	12.2	6.6	9.8	7.4	11.2	12.4	20.0
	400	0.10	4.7	3.3	5.6	4.3	6.0	3.5	19.4	16.9
		0.30	4.4	3.4	6.3	4.3	6.0	4.2	17.3	15.2
		0.50	4.7	4.7	5.9	5.7	5.6	5.0	14.6	14.2
		0.70	6.4	8.7	5.7	7.9	5.1	6.8	15.7	19.0
GH	100	0.10	4.8	2.5	5.1	2.0	7.7	2.7	15.3	11.2
		0.30	5.0	3.7	5.9	4.4	7.5	4.5	15.0	14.2
		0.50	4.5	6.7	4.3	7.1	6.3	7.9	10.7	15.7
		0.70	3.5	16.0	4.3	18.9	5.1	18.9	4.5	25.4
	200	0.10	6.4	3.9	5.6	3.7	7.3	3.9	18.2	14.1
		0.30	6.0	5.1	6.4	4.6	6.7	4.6	19.1	16.4
		0.50	5.1	4.9	6.0	6.4	6.9	8.0	15.6	17.2
		0.70	3.8	14.4	2.8	13.0	4.4	12.4	10.0	25.4
	400	0.10	5.0	4.0	5.8	4.8	6.3	5.1	18.5	16.3
		0.30	4.1	3.0	5.1	4.3	6.3	4.6	18.5	17.2
		0.50	3.2	3.6	5.0	6.3	7.9	7.5	16.7	17.2
		0.70	5.2	9.8	3.8	8.7	5.4	10.6	14.5	22.4
F	100	0.10	5.5	2.1	5.3	2.3	10.6	4.2	15.2	10.2
		0.30	4.4	2.2	5.9	3.9	7.7	4.1	13.3	10.3
		0.50	4.0	7.6	4.0	6.0	5.4	7.1	12.8	18.0
		0.70	5.2	29.3	4.8	26.5	5.4	18.1	5.9	28.5
	200	0.10	4.0	2.1	6.0	3.9	8.3	4.5	17.5	13.4
		0.30	5.0	3.9	5.7	4.1	7.1	3.9	17.0	14.5
		0.50	4.8	6.2	4.5	5.7	6.9	7.1	15.0	17.3
		0.70	3.2	19.9	4.0	17.5	4.6	13.4	8.9	25.1
	400	0.10	4.1	3.1	6.0	4.4	6.0	4.0	18.0	14.8
		0.30	5.5	4.6	6.7	5.6	5.9	4.2	14.7	12.5
		0.50	4.6	4.7	4.7	5.0	4.0	3.8	15.7	16.5
		0.70	5.3	13.2	4.5	12.3	6.2	9.9	14.2	21.7

Table 4: Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{100, 200\}$ generated with $\beta \in \{0, 0.5\}$, $t \in \{0.1, 0.25, 0.5\}$ and when C_1 and C_2 are both bivariate Clayton (Cl), Gumbel–Hougaard (GH), normal (N) or Frank (F) copulas with a Kendall's tau of 0.2 for C_1 and a Kendall's tau of τ for C_2. The colunms CvM give the results for the test studied in Bücher et al. (2014). The latter test and the test $\tilde{S}_{n,1}$ (resp. the test $S^a_{n,1}$) are (resp. is) carried out using dependent multiplier sequences (resp. a variance estimator of the form (3.12)).

				$\beta = 0$			$\beta = 0.5$							$\beta = 0$			$\beta = 0.5$		
C	n	τ	t	CvM	$\tilde{S}_{n,1}$	$S^a_{n,1}$	CvM	$\tilde{S}_{n,1}$	$S^a_{n,1}$	C	n	τ	t	CvM	$\tilde{S}_{n,1}$	$S^a_{n,1}$	CvM	$\tilde{S}_{n,1}$	$S^a_{n,1}$
Cl	100	0.4	0.10	6.5	6.5	4.3	6.5	8.0	5.0	N	100	0.4	0.10	6.1	7.8	6.2	6.9	10.2	7.8
			0.25	17.9	20.4	13.4	14.0	19.7	10.6				0.25	14.4	19.3	14.7	13.7	19.2	13.2
			0.50	23.5	23.2	15.0	18.3	22.4	9.7				0.50	25.6	27.7	19.4	17.5	24.1	12.5
		0.6	0.10	12.6	20.6	19.7	9.4	17.1	17.0			0.6	0.10	10.6	27.1	32.0	8.2	19.7	23.7
			0.25	61.3	65.7	52.7	44.2	53.6	36.4				0.25	61.5	70.1	61.3	46.0	62.3	44.8
			0.50	80.0	78.8	61.1	58.4	61.8	34.9				0.50	82.6	85.1	72.3	64.9	71.3	44.9
	200	0.4	0.10	8.2	9.6	7.5	6.9	10.4	7.0		200	0.4	0.10	8.0	10.8	9.2	5.9	12.6	9.2
			0.25	26.5	31.8	25.2	19.9	27.7	20.2				0.25	27.7	37.4	33.2	20.4	31.0	24.7
			0.50	45.3	47.0	37.0	34.2	40.0	27.9				0.50	47.0	51.5	43.6	33.2	41.7	30.7
		0.6	0.10	30.4	42.1	42.3	12.6	28.8	28.6			0.6	0.10	27.1	47.3	49.6	14.5	35.6	39.2
			0.25	93.2	94.2	87.4	71.1	79.2	65.9				0.25	91.5	96.5	88.4	72.3	85.2	71.0
			0.50	98.5	98.3	94.1	89.5	90.5	80.1				0.50	98.8	99.7	96.3	91.7	95.5	83.6
GH	100	0.4	0.10	5.3	8.0	7.1	5.0	8.2	7.1	F	100	0.4	0.10	6.2	8.6	6.9	5.5	9.6	8.6
			0.25	12.4	17.1	12.1	11.6	18.6	11.1				0.25	14.7	20.0	15.0	13.1	20.2	14.2
			0.50	22.5	25.2	16.9	18.2	24.2	14.0				0.50	28.4	29.9	21.8	19.9	25.8	15.2
		0.6	0.10	10.4	18.5	26.1	7.7	19.4	25.7			0.6	0.10	11.4	24.2	30.9	8.8	19.1	24.9
			0.25	53.3	63.1	54.7	41.2	58.0	43.7				0.25	63.6	72.6	62.0	42.4	57.3	38.2
			0.50	78.1	80.4	67.4	62.7	69.5	46.1				0.50	83.2	85.4	71.4	62.4	68.6	42.9
	200	0.4	0.10	7.0	10.5	10.0	7.1	11.4	9.9		200	0.4	0.10	8.5	12.9	11.6	5.3	9.2	6.6
			0.25	25.2	31.9	27.7	19.1	30.9	22.8				0.25	31.3	39.5	36.7	18.7	28.3	21.2
			0.50	43.0	48.3	42.1	31.4	39.3	30.0				0.50	49.9	55.0	47.5	36.1	41.6	32.1
		0.6	0.10	25.9	42.7	47.2	13.0	30.1	34.0			0.6	0.10	30.0	49.4	53.2	13.0	32.3	36.1
			0.25	89.0	92.9	86.3	72.1	83.5	70.0				0.25	94.4	97.4	85.4	74.9	84.3	71.5
			0.50	98.3	98.5	95.9	89.6	92.0	83.4				0.50	99.1	99.5	95.6	91.0	93.6	80.9

Table 5: Percentage of rejection of H_0 computed from 1000 samples of size $n = 500$ generated with $\beta = 0$ and when C_1 and C_2 are both either the bivariate Student copula with 1 d.f. (t_1), with 3 d.f. (t_3) or with 5 d.f. (t_5) with a Spearman's rho of 0.4 for C_1 and a Spearman's rho of ρ for C_2. The test $\tilde{S}_{n,1}$ was carried out with dependent multiplier sequences, while the test $S^a_{n,1}$ used a variance estimator of the form (3.12). The columns W contain the rejection rates of the similar test studied in Wied et al. (2014). The results are taken from Table 1 in the latter reference.

	t_1			t_3			t_5		
ρ	W	$\tilde{S}_{n,1}$	$S^a_{n,1}$	W	$\tilde{S}_{n,1}$	$S^a_{n,1}$	W	$\tilde{S}_{n,1}$	$S^a_{n,1}$
0.4	4.5	3.9	2.8	4.5	5.2	4.0	4.7	6.3	4.4
0.6	8.1	43.3	38.7	8.5	57.9	54.3	8.5	66.5	63.8
0.8	20.5	99.4	98.6	21.7	100.0	99.9	21.5	100.0	100.0
0.2	7.9	33.7	29.2	8.8	51.0	46.6	8.9	52.9	48.4
0.0	19.9	87.7	84.7	23.0	95.7	94.9	24.0	97.2	96.3
-0.2	41.8	99.7	99.6	49.5	100.0	100.0	51.5	100.0	100.0
-0.4	70.2	100.0	100.0	78.6	100.0	100.0	80.4	100.0	99.9
-0.6	91.7	100.0	99.9	95.8	100.0	100.0	96.6	100.0	100.0

Conclusion

Dans ce travail de thèse, nous avons considéré des observations $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ multivariées, potentiellement sériellement dépendantes et dont les f.d.r. marginales sont supposées continues. Nous avons proposé deux nouvelles procédures non paramétriques pour tester l'hypothèse $\mathcal{H}_0$ suivante :

$$\mathcal{H}_0 : \quad \text{Il existe } F \text{ telle que } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n \text{ admettent tous } F \text{ pour f.d.r.}$$

Dans la première contribution de ce travail de thèse correspondant à l'article de la section 3.2, nous avons proposé un test qui semble plus puissant que ses prédécesseurs pour l'alternative $\mathcal{H}_{0,m} \cap \neg(\mathcal{H}_{0,c})$, où $\mathcal{H}_{0,m}$ et $\mathcal{H}_{0,c}$ sont définies par :

$$\mathcal{H}_{0,c} : \quad \exists C \text{ telle que } C \text{ est la copule des vecteurs aléatoires } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n,$$
$$\mathcal{H}_{0,m} : \quad \exists F_1, \dots, F_d \text{ telles que } F_1, \dots, F_d \text{ sont les f.d.r. marginales de } \boldsymbol{X}_1, \dots, \boldsymbol{X}_n.$$

D'un point de vue théorique, nous avons établi la validité asymptotique sous $\mathcal{H}_0$ d'une procédure de rééchantillonnage de la statistique de test, à base de *multiplicateurs*. Bien que cette dernière ne puisse pas de mettre sous la forme *cumulative sum*, sa construction reprend néanmoins l'idée générale de cette approche. C'est une statistique à la Cramér-von Mises, construite à partir d'un processus pouvant s'écrire comme la différence pondérée entre la copule empirique calculée sur une première partie de l'échantillon et la copule empirique calculée sur le reste de l'échantillon.

Les résultats de cette première contribution sont démontrés sous une condition de régularité portant sur les dérivées partielles de la copule. Nous nous sommes cependant passés de la condition qui suppose qu'il n'y a pas d'ex-æquo presque sûrement dans les j^e composantes des observations, pour tout $j \in \{1, \dots, d\}$. Les simulations de Monte Carlo effectuées semblent indiquer que la nouvelle stratégie de rééchantillonnage conduit à de meilleures performances en termes de puissance que le rééchantillonnage de Bücher et Ruppert (2013) ou celui suggéré dans Bücher et Kojadinovic (2013).

La deuxième contribution de cette thèse, correspondant à l'article de la section 4.3, a consisté à proposer des tests plus puissants que celui de la première contribution, pour l'alternative $\mathcal{H}_{0,m} \cap \neg(\mathcal{H}_{0,c})$, dans le cas d'une rupture dans le *rho de Spearman multivarié*. Ces tests reprennent là encore, la même idée générale, issue de l'approche *cumulative sum*. Par construction, ils présentent le désavantage de ne pas être consistants pour $\mathcal{H}_0$. Cependant, contrairement aux résultats théoriques de la première contribution, les résultats relatifs à ces tests présentent l'avantage de ne pas requérir de conditions de régularité sur les dérivées partielles de la copule.

Pour la mise en œuvre de ces tests fondés sur le rho de Spearman, nous avons comparé deux stratégies. La première utilise une méthode de rééchantillonage de la statistique à base de multiplicateurs. L'autre est fondée sur une estimation de la loi asymptotique de la statistique sous $\mathcal{H}_0$. D'une façon générale, les versions des tests fondées sur le rééchantillonage se comportent mieux. Une alternative intéressante à la seconde méthode serait de considérer une version des tests fondée sur l'approche «*self-normalization*» que l'on retrouve dans SHAO et ZHANG (2010).

Comme on pouvait s'y attendre, les simulations de Monte Carlo révèlent que les tests étudiés dans la seconde contribution présentent globalement de meilleurs résultats à la fois en termes de puissance, sous des alternatives $\mathcal{H}_{0,m} \cap \neg(\mathcal{H}_{0,c})$ comportant un changement dans le rho de Spearman, et en termes de temps de calcul.

Il est important de garder à l'esprit que les tests proposés dans l'ensemble de ce travail sont des tests uniquement pour $\mathcal{H}_0$ et non pour $\mathcal{H}_{0,c}$. En effet, lorsque le calcul de la valeur p de l'un de ces tests amène à rejeter l'hypothèse $\mathcal{H}_0$, on ne pourra conclure en faveur d'une rupture dans la copule des observations qu'en faisant l'hypothèse supplémentaire $\mathcal{H}_{0,m}$ qu'il n'y a pas de rupture dans une des marges des observations. Cette hypothèse peut être testée en considérant les échantillons $X_{1j}, \dots, X_{nj}$, pour $j \in \{1, \dots, d\}$, et en utilisant par exemple, la procédure décrite dans HOLMES et coll. (2013) lorsque les observations sont sériellement indépendantes. En cas de rejet d'au moins un des tests portant sur les marges, on ne pourra pas conclure en faveur d'une rupture dans la copule des observations. C'est pourquoi il serait intéressant de mettre en place un test directement pour $\mathcal{H}_{0,c}$.

De plus, un rejet de l'hypothèse $\mathcal{H}_0$ ne permet pas d'obtenir directement de l'information sur la nature de la (ou des) rupture(s) : il peut s'agir d'une ou plusieurs ruptures abruptes et/ou graduelles.

D'autres directions de recherche liées aux travaux proposés dans les articles des sections 3.2 et 4.3 sont envisageables. Nous pourrions par exemple étudier, dans l'esprit de Lombard (1987) et Harel et Puri (1990), le comportement asymptotique sous $\mathcal{H}_0$ de processus basés sur d'autres statistiques de rangs, au travers de fonctions scores J sur $]0,1[^d$. Les choix des fonctions scores

$$J_1(\boldsymbol{u}) = \prod_{j=1}^{d}(1-u_j), \quad J_2(\boldsymbol{u}) = \prod_{j=1}^{d} u_j, \quad J_3(\boldsymbol{u}) = \binom{2}{d}^{-1} \sum_{1<p<q<d}(1-u_p)(1-u_q),$$

amènent en particulier à la construction des statistiques $S_{n,i}$, $i = 1, 2, 3$ définies dans l'équation (2.3) de l'article de la section 4.3. Dans le cas de la dimension 2, une telle méthodologie permettrait d'adapter nos résultats à des statistiques classiques utilisées dans la littérature, comme les statistiques de van der Waerden (Fisher–Yates), exponentielles, de Blest ou de Wilcoxon en utilisant respectivement les fonctions scores définies pour $(u,v) \in]0,1[^2$ par $J_{vdW}(u,v) = \Phi^{-1}(u)\Phi^{-1}(v)$, où Φ^{-1} désigne la fonction quantile de la loi normale centrée réduite, $J_{exp}(u,v) = (1+\log(1-u))(1+\log(1-v))$, $J_{Bl}(u,v) = \{1-3(1-u)^2\}(2v-1)$ et $J_{Wil}(u,v) = (2u-1)\log\{v/(1-v)\}$. En se fondant sur les résultats des simulations de la section 4 de l'article de la section 4.3, il pourrait être judicieux d'utiliser un rééchantillonnage à base de multiplicateurs, similaire à ceux de l'article en question.

Ces considérations constituent nos perspectives de recherche prioritaires.

Bibliographie

BAI, J. 1994, «Weak convergence of the sequential empirical processes of residuals in ARMA models», The Annals of Statistics, vol. 22, nº 4, p. 2051–2061. 34

BÜCHER, A. 2013, «A note on weak convergence of the sequential multivariate empirical process under strong mixing», Journal of Theoretical Probability, p. in press. 21, 22, 27

BÜCHER, A. et H. DETTE. 2010, «A note on bootstrap approximations for the empirical copula process», Statistics and Probability Letters, vol. 80, nº 23–24, p. 1925–1932. 23

BÜCHER, A. et I. KOJADINOVIC. 2013, «A dependent multiplier bootstrap for the sequential empirical copula process under strong mixing», arXiv :1306.3930. 3, 18, 21, 24, 26, 27, 29, 30, 31, 111

BÜCHER, A. et M. RUPPERT. 2013, «Consistent testing for a constant copula under strong mixing based on the tapered block multiplier technique», Journal of Multivariate Analysis, vol. 116, p. 208–229. 3, 24, 26, 29, 31, 111

BÜCHER, A. et J. SEGERS. 2013, «Extreme value copula estimation based on block maxima of a multivariate stationary time series», arXiv :1311.3060. 18, 19

BÜCHER, A. et S. VOLGUSHEV. 2013, «Empirical and sequential empirical copula processes under serial dependence», Journal of Multivariate Analysis, vol. 119, p. 61–70. 19, 21, 22

BÜHLMANN, P. 1993, The blockwise bootstrap in time series and empirical processes, thèse de doctorat, ETH Zürich. Diss. ETH No. 10354. 29

CSÖRGŐ, M. et L. HORVÁTH. 1997, Limit theorems in change-point analysis, Wiley Series in Probability and Statistics, John Wiley & Sons, Chichester, UK. 2, 33

Deheuvels, P. 1979, «La fonction de dépendance empirique et ses propriétés : un test non paramétrique d'indépendance», Acad. Roy. Belg. Bull. Cl. Sci. 5th Ser., vol. 65, p. 274–292. 2, 18

Deheuvels, P. 1981, «A non parametric test for independence», Publications de l'Institut de Statistique de l'Université de Paris, vol. 26, p. 29–50. 18

Dehling, H. et W. Philipp. 2002, «Empirical process techniques for dependent data», dans Empirical process techniques for dependent data, édité par H. Dehling, T. Mikosch et M. Sorensen, Birkhäuser, Boston, p. 1–113. 22

Dehling, H., D. Vogel, M. Wendler et D. Wied. 2014, «An efficient and robust test for a change-point in correlation», arXiv :1203.4871v3. 3

Durante, F. et C. Sempi. 2010, «Copula theory : An introduction», dans Copula theory and its applications (Warsaw, 2009), édité par P. Jaworski, F. Durante, W. Härdle et W. Rychlik, Lecture Notes in Statistics, Springer-Verlag, p. 3–32. 5

Fermanian, J.-D., D. Radulovic et M. Wegkamp. 2004, «Weak convergence of empirical copula processes», Bernoulli, vol. 10, n° 5, p. 847–860. 18, 19, 20, 22, 29

Gänssler, P. et W. Stute. 1987, Seminar on empirical processes, DMV Seminar 9, Birkhäuser, Basel. 19

Gombay, E. et L. Horváth. 1999, «Change-points and bootstrap», Environmetrics, vol. 10, n° 6. 2, 3, 34

Gombay, E. et L. Horváth. 2002, «Rates of convergence for U-statistic processes and their bootstrapped versions», Journal of Statistical Planning and Inference, vol. 102, p. 247–272. 3

Hájek, J., Z. Šidák et P. K. Sen. 1967, Theory of rank tests, Academic press New York. 17

Harel, M. et M. L. Puri. 1990, «Weak convergence of the serial linear rank statistic with unbounded scores and regression constants under mixing conditions», Journal of Statistical Planning and Inference, vol. 25, n° 2, p. 163–186. 113

Hoeffding, W. 1940, Maszstabinvariante Korrelationstheorie, Teubner. S, Leipzig, 1 vol. (181-233 p.) p.. URL http://www.sudoc.fr/111704804, edition de Dissertation Wissenschaften Berlin 1940. 12

Holmes, M., I. Kojadinovic et J.-F. Quessy. 2013, «Nonparametric tests for change-point detection à la Gombay and Horváth», Journal of Multivariate Analysis, vol. 115, p. 16–32. 2, 28, 34, 35, 112

Inoue, A. 2001, «Testing for distributional change in time series», Econometric Theory, vol. 17, n° 1, p. 156–187. 2, 34

Joe, H. 1997, Multivariate models and dependence concepts, Chapman and Hall, London. 15

Kojadinovic, I., J. Segers et J. Yan. 2011, «Large-sample tests of extreme-value dependence for multivariate copulas», The Canadian Journal of Statistics, vol. 39, n° 4, p. 703–720. 24

Kosorok, M. 2008, Introduction to empirical processes and semiparametric inference, Springer, New York. 22, 28

Künsch, H. 1989, «The jacknife and the bootstrap for general stationary observations», Annals of Statistics, vol. 17, n° 3, p. 1217–1241. 29

Lombard, F. 1987, «Rank tests for changepoint problems», Biometrika, vol. 74, n° 3, p. 615–624. 1, 17, 113

McNeil, A., R. Frey et P. Embrechts. 2005, Quantitative risk management, Princeton University Press, New Jersey. 5, 8, 14

Nelsen, R. 2006, An introduction to copulas, Springer, New-York. Second edition. 5, 9, 15, 20

Pettitt, A. 1979, «A non-parametric approach to the change point problem», Applied Statistics, vol. 28, n° 2, p. 126–135. 1

Quesada-Molina, J. J. 1992, «A generalization of an identity of hoeffding and some applications», Journal of the Italian Statistical Society, vol. 1, n° 3, p. 405–411. 12

Quessy, J.-F., M. Saïd et A.-C. Favre. 2013, «Multivariate Kendall's tau for change-point detection in copulas», The Canadian Journal of Statistics, vol. 41, p. 65–82. 3, 15

R Development Core Team. 2013, R : A Language and Environment for Statistical Computing, R Foundation for Statistical Computing, Vienna, Austria. URL `http://www.R-project.org`, ISBN 3-900051-07-0. 9

Rémillard, B. 2010, «Goodness-of-fit tests for copulas of multivariate time series», Social Science Research Network, vol. 1729982, p. 1–32. URL `http://ssrn.com/abstract=1729982`. 26

Rémillard, B. et O. Scaillet. 2009, «Testing for equality between two copulas», Journal of Multivariate Analysis, vol. 100, nº 3, p. 377–386. 3, 23, 24, 29

Rosenblatt, M. 1956, «A central limit theorem and a strong mixing condition», Proceedings of the National Academy of Sciences of the United States of America, vol. 42, nº 1, p. 43. 21

Rüschendorf, L. 1976, «Asymptotic distributions of multivariate rank order statistics», Annals of Statistics, vol. 4, p. 912–923. 18, 19, 26

Scaillet, O. 2005, «A Kolmogorov-Smirnov type test for positive quadrant dependence», Canadian Journal of Statistics, vol. 33, p. 415–427. 23

Schmid, F. et R. Schmidt. 2007a, «Multivariate extensions of Spearman's rho and related statistics», Statistics & Probability Letters, vol. 77, nº 4, p. 407–416. 13, 68

Schmid, F. et R. Schmidt. 2007b, «Nonparametric inference on multivariate versions of Blomqvist's beta and related measures of tail dependence», Metrika, vol. 66, nº 3, p. 323–354. 15

Segers, J. 2012, «Asymptotics of empirical copula processes under nonrestrictive smoothness assumptions», Bernoulli, vol. 18, p. 764–782. 18, 19, 20, 23, 24

Shao, X. et X. Zhang. 2010, «Testing for change points in time series», Journal of the American Statistical Association, vol. 105, nº 491. 112

Sklar, A. 1959, «Fonctions de répartition à n dimensions et leurs marges», Publications de l'Institut de Statistique de l'Université de Paris, vol. 8, p. 229–231. 2, 5, 7

van der Vaart, A. et J. Wellner. 2000, Weak convergence and empirical processes, Springer, New York. Second edition. 16, 19, 23, 27

van der Vaart, A. et J. Wellner. 2007, «Empirical processes indexed by estimated functions», dans Asymptotics : Particles, Processes and Inverse Problems, Institute of Mathematical Statistics, p. 234–252. 19

Wied, D., H. Dehling, M. van Kampen et D. Vogel. 2014, «A fluctuation test for constant Spearman's rho with nuisance-free limit distribution», Computational Statistics and Data Analysis, vol. 76, p. 723–736. 3